Exercise #11

27. October 2025

Problem 1.

Define the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ as

$$f(x) := \begin{cases} 1 & \text{if } 0 < x < 2, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} x & \text{if } 0 < x \leq 2, \\ 4 - x & \text{if } 2 < x \leq 4, \\ 0 & \text{else.} \end{cases}$$

a) Use the definition of convolution in order to show that $f * f = g$.

b) Compute the Fourier transform of f.

c) Compute the Fourier transform of g.

Problem 2. (Fourier Transform PDE)

We consider the one-dimensional inhomogeneous heat equation on $\mathbb{R}$:

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + f(x), \qquad x \in \mathbb{R},\ t > 0,$$

with initial condition

$$u(x,0) = g(x),$$

where $f, g \in L^1(\mathbb{R})$ (absolutely integrable) are given functions. Take the Fourier transform in the spatial variable x. Denote

$$\widehat{u}(\omega,t) = \mathcal{F}_x[u(\cdot,t)](\omega) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x}\,\mathrm{d}x,$$

and similarly $\widehat{f}(\omega)$ and $\widehat{g}(\omega)$.

a) Prove the Fourier transform

$$\mathcal{F}\left(\mathrm{e}^{-\frac{1}{2}x^2}\right)(\omega) = \mathrm{e}^{-\frac{1}{2}\omega^2}.$$

Hint: First, show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$$

by considering

$$\left(\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y = \int_{\mathbb{R}^2} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y,$$

rewritten in polar coordinates. Then, show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+bx}\mathrm{d}x = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}, \quad a > 0,\ b \in \mathbb{C}.$$

b) Solve the PDE using the Fourier transform to obtain $u(x,t)$ as an integral involving the heat kernel

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Use the convolution theorem to show that

$$u(x,t) = (K(\cdot,t) * g)(x) + \int_0^t (K(\cdot,t-s) * f)(x)\,\mathrm{d}s.$$

Explain the role of the convolution theorem in constructing the solution in x. What would fail without it?
Hints: Remember that the solution of the inhomogeneous ODE $y_t = \lambda y + a$ is given by

$$y(t) = y_{\text{hom}}(t) + \int_0^t a\mathrm{e}^{\lambda(t-s)}\,\mathrm{d}s,$$

where y_{hom} is the solution of the homogeneous ODE. What is $\mathcal{F}(K(\cdot,t))$?

c) Verify directly that $\partial_t K = \partial_{xx} K$ by differentiating under the integral sign. Explain how this ensures that u indeed satisfies the PDE.
Hints: The heat kernel K satisfies $\lim_{t\to 0} K(\cdot,t) * h = h$ for a function h. It is $\frac{\mathrm{d}}{\mathrm{d}x}(h_1 * h_2)(x) = (h_1' * h_2)(x) = (h_1 * h_2')(x)$. Why?

d) Let $f(x) = e^{-x^2}$ and $g(x) = 0$. Use the formula to obtain an explicit integral expression for $u(x,t)$. Show that it can be written as

$$u(x,t) = \int_0^t \frac{1}{\sqrt{1+4\tau}} e^{-\frac{x^2}{1+4\tau}} \,\mathrm{d}\tau.$$

What is the solution for $f(x) = 0$, $g(x) = e^{-x^2}$?

e) Discuss the effect of f in the long-time behaviour $t \to \infty$. Consider $\hat{f}(0) = 0$ and $\hat{f}(0) \neq 0$.

Problem 3.

a) Let $w_N := \mathrm{e}^{-\frac{2\pi \mathrm{i}}{N}}$ for some $N \in \mathbb{N}$. Show that,

$$\sum_{k=0}^{N-1} w_N^{jk} = \begin{cases} N, & \text{if } j \bmod N = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $j \in \mathbb{Z}$ is some integer.

b) Show that,

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i n}{N}k} e^{-\frac{2\pi i m}{N}k} = \begin{cases} N, & n = m, \\ 0, & \text{otherwise,} \end{cases}$$

for some $n, m \in \{0, \dots, N-1\}$.

c) Compute explicitly the Fourier matrices $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$.

d) Compute by hand the discrete Fourier transform of the vector $f = (2, 3, 1, 2)$.

e) Assume that the discrete Fourier transform of the vector $f \in \mathbb{R}^{11}$ is given as

$$\hat{f} = (0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0).$$

Express the entries f_n of f in terms of sine and cosine functions.

Problem 4. (Signal filtering with DFT)

A mechanical engineer is measuring the vibrations of a dynamic system. After the experiment, she sees that the results look very "shaky", which is unexpected. Then she notices her mistake: she left the fan on, and too close to the testing platform. The fan's vibrations ended up contaminating her measurements, but she has an idea to fix this without having to re-do that expensive experiment. A quick internet search tells her most table fans rotate with at least 10 Hz, and from Physics she knows that her dynamic system should not vibrate faster than 5 Hz. Therefore, she can use a discrete Fourier transform (DFT) to filter out high frequencies. To help with this task, we have created a Jupyter notebook $\mathrm{DFT.ipynb}$. The idea is to compute the DFT of the signal, remove from the spectrum any frequency above a cut-off value k_{cut}, then compute the filtered signal through the *inverse* DFT.

a) From the frequency spectrum plotted on the Jupyter notebook, answer: what are the three frequencies that compose the original (unfiltered) signal?

b) Which of these frequencies is the dominant one in the measured signal?

c) What is the fan's frequency, in rotations per minute (rpm)?

d) Based on the engineer's prior knowledge, choose an appropriate cut-off frequency k_{cut} to filter the measurements, and plot the original and filtered signals, together. What is the frequency of the dynamic system (filtered signal)?
Hint: there isn't only one acceptable value for k_{cut}. You just have to be sure to take it within an *appropriate* interval.

The next exercises are optional and should not be handed in!

Problem 5.

Compute the Fourier transform of the following functions $f : \mathbb{R} \to \mathbb{R}$:

a) The function
$$f(x) = xe^{-|x|}.$$

b) The function
$$f(x) = \begin{cases} \sin(x) & \text{if } 0 < x < \pi, \\ 0 & \text{else.} \end{cases}$$

Problem 6.

a) Compute the Fourier transform of $f(t)$ where,
$$f(t) = \begin{cases} 1, & t \in [-a, a] \\ 0, & \text{otherwise,} \end{cases}$$
for some $a > 0$. (Note that this was done in the lecture for a variable interval $[a, b]$, but in this problem you should compute the integral for yourself).

b) Compute the Fourier transform $f(t)$ where,
$$f(t) = t^2 \mathrm{e}^{-t^2}.$$
Hint: Differentiate e^{-t^2} twice.

c) Compute the Fourier transform of $f(t)$, where
$$f(t) = \mathrm{e}^{-|t|}.$$

Problem 7.

Compute the convolution $f * f$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the function

$$f(x) = \operatorname{sinc}(x) := \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Hint: Use the Fourier transform!

Problem 8. (Discrete Fourier Transform)

For the signal $f = \left(\frac{1}{2}, 1, \frac{1}{2}, 0\right)^T \in \mathbb{R}^4$ we want to consider the discrete Fourier transform $\hat{f} = \mathcal{F}_4 f$.

a) What does the matrix $\mathcal{F}_4$ look like?

b) Compute $\hat{f}$.

c) Let $c \in \mathbb{R}$ be given and assume that for another signal g we obtain $\hat{g} = (\hat{g}_0, \hat{g}_1, \hat{g}_2, \hat{g}_3) = \mathcal{F}_4 g$ with $\hat{g}_1 = \hat{g}_3 = c$ and $\hat{g}_0 = \hat{g}_2 = 0$. What is the simplest function $g(x)$ that could have been sampled?
Hint: Think of a bandlimited function or a trigonometric polynomial $g(x)$.

d) Is the inverse Fourier transform $h = \mathcal{F}_8^{-1}\hat{h}$ of $\hat{h} = (0, 0, 0, 0, 0, 0, 1, 0)$ real-valued?

Problem 9. (Discrete Fourier Transform, J)

a) Implement a python function to construct the Fourier matrix $\mathcal{F}_N$ for an input N.

b) Using that, compute the discrete Fourier transform of the signal

$$f = (3, 2, -1, 1, -1, 2, 0, 3, 1, -2)^T.$$

c) Consider the function $f(t) = \tan\big(sin(2\pi t)\big)$. Sample its values f_n using a sampling frequency of 100 Hz ($\Delta t = 0.01$ s between samples) in the interval $[0, 2]$. Then, compute the discrete Fourier transform $\hat{f}$ of the sampled signal and plot the spectrum (plot $\frac{|\hat{f}|}{N}$ against $\frac{n}{N\Delta t}$) for $n = 0, 1, \ldots, \lfloor \frac{N}{2} \rfloor$, where $\lfloor x \rfloor$ is x rounded down (floor(x)), (there is no need to plot the second half of the spectrum since it will be simply mirrored). How many peaks are clearly visible, or in other words, how many nodes are actually significant? At what frequencies are these peaks?

d) Now filter out all entries $\hat{f}_n$ of $\hat{f}$, whose "power density" $\frac{|\hat{f}_n|}{N}$ is less than 0.05, reconstruct the filtered signal and compare it to the unfiltered one, plotting them together.
Remember: $\mathcal{F}_N^{-1} = \frac{1}{N}\overline{\mathcal{F}_N}$

Problem 10. (FFT low pass filter - J)

While conducting an experiment there was an interfering signal. Thankfully the researchers know that the interfering signal had frequencies higher than 8 Hz, while their experimental data should only contain lower frequencies. Pretend the measured data are samples from

$$f(t) = \sin^3(2\pi t) + \cos(6\pi t)\cos(12\pi t)\sin(6\pi t)$$

with sample resolution $\Delta t = 0.001$ in the interval $[0, 4]$.

a) Plot the experimental data.

b) Compute the discrete Fourier transform of the data and plot the spectrum (plot $|\hat{f}_n|/N$ against $n/(N\Delta t)$) for $n = 0, 1, \ldots, \lfloor N/2 \rfloor$, where $\lfloor x \rfloor$ is x rounded down (floor) (there is no need to plot the second half of the spectrum since it will simply be mirrored).

c) Now filter out all entries of $\hat{f}$, whose frequency is higher than 8 Hz, compute the inverse Fourier transform of the filtered data and plot the data and the filtered data together.

Problem 11.

Use the Fourier transform to find the function $u(x,t)$ satisfying the *telegraph equation*

$$u_{tt} + 2u_t + u = u_{xx} \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0$$

with the initial conditions

$$\left.\begin{aligned} u(x,0) &= \operatorname{sinc}(x), \\ \partial_t u(x,0) &= -\operatorname{sinc}(x) \end{aligned}\right\} \qquad \text{for } x \in \mathbb{R}.$$

Here the sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$