exercise 3

these are my solutions to the third exercise set of TMA4135.

there should be a python source code file attached to this deliverable.

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problem 1

a) & b)

using the trapezoidal rule we can estimate

$$(1) \quad I_{[0,1]}[\exp] := \int_0^1 e^x \, dx, \qquad (2) \quad I_{[0,1]}[\ln] := \int_0^1 \ln(x) \, dx$$

let us solve both analytically first, starting with (1)

$$\int_0^1 e^x \, dx = e^x|_0^1 = e - 1$$

then (2)

$$\begin{aligned} \int_0^1 \ln(x) \, dx &= [x \ln(x) - x]_0^1 \\ &= (1 \ln(1) - 1) - \lim_{t \to 0^+} (t \ln(t) - t) \\ &= -1 - 0 = -1, \end{aligned}$$

where we used integration by parts and the fact that $\lim_{t \to 0^+} t \ln(t) = 0$.

now we know what to expect from our estimates.

recall the trapezoidal rule

$$I_{[a,b]}[f] \approx T_{[a,b]}[f] := \frac{b-a}{2}(f(a) + f(b))$$

in our case $a = 0$ and $b = 1$. for (1) with $f(x) = e^x$ we obtain

$$T_{[0,1]}[\exp] = \frac{1}{2}\left(e^0 + e^1\right) = \frac{1}{2}(1+e) = \frac{1+e}{2},$$

which yields an error of

$$E_{[0,1]}[\exp] = (e-1) - \frac{1+e}{2} = \frac{2e-2-1-e}{2} = \frac{e-3}{2}.$$

for (2) with $f(x) = \ln(x)$ we obtain

$$T_{[0,1]}[\ln] = \frac{1}{2}(\ln(0) + \ln(1)) = \frac{1}{2}(-\infty + 0) = -\infty$$

which yields an error of $+\infty$.

thus we can see that using the trapezoidal rule for $\ln(x)$ diverges and proves to be unwieldy. for $\exp(x)$ the actual error is $\frac{e-3}{2} \approx -0.141$.

c)

recall the upper bound of the error for the trapezoidal rule

$$\left|E_{[a,b]}[f]\right| \leqslant \frac{(b-a)^3}{12} \cdot \max_{\xi \in [a,b]} |f''(\xi)|$$

which is for $f = \exp$ on $[0,1]$

$$\left|E_{[0,1]}[\exp]\right| \leqslant \frac{1^3}{12} \cdot \max_{\xi \in [0,1]} \left|e^{\xi}\right| = \frac{e}{12} \approx 0.227$$

Our actual error magnitude is $\left|\frac{e-3}{2}\right| = \frac{3-e}{2} \approx 0.141$.

Since $0.141 \leqslant 0.227$, the error bound is satisfied.

d)

recall the error bound derived in the lecture,

$$\left|E_{[0,1]}[f]\right| \leqslant \frac{\mathrm{max}_{\xi \in [a,b]}\left|f^{(n+1)}(\xi)\right|}{(n+1)!} \int_a^b \prod_{i=0}^{n} |x - x_i| \, dx,$$

where the integral of f is taken over the interval $[a, b]$, and $f \in C^{n+1}[0, 1]$.

for the trapezoidal rule, $n = 1$, so we have $x_0 = 0$ and $x_1 = 1$. for $f(x) = e^x$:

- $f^{(n+1)}(x) = f^{(2)}(x) = e^x$
- $\mathrm{max}_{\xi \in [0,1]}\left|e^{\xi}\right| = e$
- $(n+1)! = 2! = 2$

the integral becomes:

$$\begin{aligned}
\int_0^1 \prod_{i=0}^{1} |x - x_i| \, dx &= \int_0^1 |x - 0| \cdot |x - 1| \, dx \\
&= \int_0^1 x \cdot |x - 1| \, dx \\
&= \int_0^1 x \cdot (1 - x) \, dx \quad \mathrm{since}\ x - 1 \leqslant 0\ \mathrm{on}\ [0, 1] \\
&= \int_0^1 \left(x - x^2\right) dx \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

therefore, the error bound is:

$$\left|E_{[0,1]}[\exp]\right| \leqslant \frac{e}{2} \cdot \frac{1}{6} = \frac{e}{12} \approx 0.227$$

this gives the same bound as part c), confirming that both error bound formulas are equivalent for the trapezoidal rule. the actual error $\frac{3-e}{2} \approx 0.141$ is indeed less than this bound.

e)

to derive the composite trapezoidal rule, we divide the interval $[a, b]$ into n segments of equal length $h = \frac{b-a}{n}$, with points $x_i = a + ih$ for $i = 0, 1, ..., n$.

applying the basic trapezoidal rule to each subinterval $[x_i, x_{i+1}]$:

$$T_{[x_i,x_{i+1}]}[f] = \frac{h}{2}(f(x_i) + f(x_{i+1}))$$

the composite rule sums over all subintervals:

$$T^n_{[a,b]}[f] = \sum_{i=0}^{n-1} T_{[x_i,x_{i+1}]}[f] = \sum_{i=0}^{n-1} \frac{h}{2}(f(x_i) + f(x_{i+1}))$$

expanding this sum:

$$T^n_{[a,b]}[f] = \frac{h}{2}[f(x_0) + f(x_1) + f(x_1) + f(x_2) + ... + f(x_{n-1}) + f(x_n)]$$

we observe that $f(x_0)$ and $f(x_n)$ appear once, while $f(x_1), f(x_2), ..., f(x_{n-1})$ each appear twice. collecting terms:

$$T^n_{[a,b]}[f] = \frac{h}{2}(f(x_0) + f(x_n)) + h\sum_{i=1}^{n-1} f(x_i)$$

since $x_0 = a$ and $x_n = b$, this gives us the desired composite formula.

f)

to derive the error bound for the composite trapezoidal rule, we start with the error bound for each subinterval.

for each subinterval $[x_i, x_{i+1}]$ with length h:

$$|E_{[x_i,x_{i+1}]}[f]| \leqslant \frac{h^3}{12} \max_{\xi \in [x_i,x_{i+1}]} |f''(\xi)|$$

the total error for the composite rule is:

$$|E^n_{[a,b]}[f]| = |\sum_{i=0}^{n-1} E_{[x_i,x_{i+1}]}[f]|$$

using the triangle inequality:

$$|\sum_{i=0}^{n-1} E_{[x_i,x_{i+1}]}[f]| \leqslant \sum_{i=0}^{n-1} |E_{[x_i,x_{i+1}]}[f]|$$

substituting the error bound for each subinterval:

$$\sum_{i=0}^{n-1} |E_{[x_i,x_{i+1}]}[f]| \leqslant \sum_{i=0}^{n-1} \frac{h^3}{12} \max_{\xi \in [x_i,x_{i+1}]} |f''(\xi)|$$

since $[x_i, x_{i+1}] \subset [a, b]$, we have:

$$\max_{\xi \in [x_i,x_{i+1}]} |f''(\xi)| \leqslant \max_{\xi \in [a,b]} |f''(\xi)|$$

therefore:

$$\sum_{i=0}^{n-1} \frac{h^3}{12} \max_{\xi \in [x_i, x_{i+1}]} |f''(\xi)| \leqslant \frac{h^3}{12} \sum_{i=0}^{n-1} \max_{\xi \in [a,b]} |f''(\xi)|$$
$$= \frac{h^3}{12} \cdot n \cdot \max_{\xi \in [a,b]} |f''(\xi)|$$
$$= h^2 \cdot \frac{b-a}{12} \max_{\xi \in [a,b]} |f''(\xi)|$$
$$= \frac{(b-a)^3}{12n^2} \max_{\xi \in [a,b]} |f''(\xi)|$$

g)

using the error bound formula:

$$|E^n_{[a,b]}[f]| \leqslant \frac{(b-a)^3}{12n^2} \max_{\xi \in [a,b]} |f''(\xi)|$$

first problem: $|E^n_{[0,1]}[\exp]| \leqslant 10^{-3}$

for $f(x) = e^x$, we have $f''(x) = e^x$. since e^x is increasing, $\max_{\xi \in [0,1]} |f''(\xi)| = e$.

$$\frac{1^3}{12n^2} \cdot e \leqslant 10^{-3}$$

solving: $n^2 \geqslant \frac{e}{12 \cdot 10^{-3}} = 1000 \frac{e}{12}$

therefore $n \geqslant \sqrt{1000 \frac{e}{12}} \approx 15.05$, so $n \geqslant 16$.

second problem: $|E^n_{[0,1]}[\ln]| \leqslant 10^{-5}$

for $f(x) = \ln(x)$, we have $f''(x) = -\frac{1}{x^2}$. since $|f''(x)| = \frac{1}{x^2} \to \infty$ as $x \to 0^+$, the function is not twice differentiable at $x = 0$. the trapezoidal rule error bound does not apply.

third problem: $|E_{[1,2]}^{n}\left[\frac{e^x}{x}\right]| \leqslant 10^{-3}$

for $f(x) = \frac{e^x}{x}$, using the quotient rule: $f''(x) = \frac{e^x(x^2-2x+2)}{x^3}$

to find the maximum, we check:
- $f''(1) = e$
- $f''(1.5) \approx 1.66$
- $f''(2) = \frac{e^2}{4} \approx 1.85$

since $f''(1) > f''(1.5)$ and $f''(1) > f''(2)$, we have $\max_{\xi \in [1,2]} |f''(\xi)| = e$.

$$\frac{1^3}{12n^2} \cdot e \leqslant 10^{-3}$$

solving: $n \geqslant \sqrt{1000\frac{e}{12}} \approx 15.05$, so $n \geqslant 16$.

h) & i)

empirically testing shows that $n = 12$ subintervals already achieves $|E_{[0,1]}^{12}[\exp]| \leqslant 10^{-3}$. furthermore, we need 120 subintervals for error 10^{-5}.

j)

to estimate the convergence rate, we fit the model $E_n \approx Ch^p$ by taking logarithms:

$$\log|E_n| \approx \log C + p \log h$$

using linear regression on $(\log h, \log|E_n|)$ data for various values of n:

for $\exp(x)$ on $[0, 1]$:
- estimated $p_1 \approx 2.0$

for $\sqrt{x}$ on $[0, 1]$:

- estimated $p_2 \approx 1.5$

k)

for exp(x): $p_1 \approx 2.0 \Rightarrow$ quadratic

for sqrt(x): $p_2 \approx 1.5$ shows that $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ is unbounded as $x \to 0^+$.

l)

given

$$E^1_{[a,b]} = -\frac{M}{12}(b-a)^3 \quad \text{and} \quad E^2_{[a,b]} = -\frac{M}{48}(b-a)^3$$

we obtain $E^1_{[a,b]} = 4E^2_{[a,b]}$.

since

$$E^1_{[a,b]} = I_{[a,b]}[f] - T^1_{[a,b]} \quad \text{and} \quad E^2_{[a,b]} = I_{[a,b]}[f] - T^2_{[a,b]}$$

we get

$$E^1_{[a,b]} - E^2_{[a,b]} = T^2_{[a,b]} - T^1_{[a,b]}$$

thus

$$\begin{aligned} 4E^2_{[a,b]} - E^2_{[a,b]} &= T^2_{[a,b]} - T^1_{[a,b]} \\ 3E^2_{[a,b]} &= T^2_{[a,b]} - T^1_{[a,b]} \\ \Longrightarrow E^1_{[a,b]} &= 4E^2_{[a,b]} = \frac{4}{3}\Big(T^2_{[a,b]} - T^1_{[a,b]}\Big) =: \mathcal{E}^1_{[a,b]}[f] \\ E^2_{[a,b]} &= \frac{1}{3}\Big(T^2_{[a,b]} - T^1_{[a,b]}\Big) =: \mathcal{E}^2_{[a,b]}[f] \end{aligned}$$

m)

for $f(x) = \sqrt{x}$ on $[0, 1]$ with 10 uniform intervals and tolerance 10^{-4}:

Interval	$T^1_{[a,b]}$	$T^2_{[a,b]}$	$\lvert\mathcal{E}^1_{[a,b]}\rvert$
$[0, 0.1]$	0.015811	0.019086	0.004367
$[0.1, 0.2]$	0.038172	0.038451	0.000372
$[0.2, 0.3]$	0.049747	0.049873	0.000168
$[0.3, 0.4]$	0.059009	0.059085	0.000101
$[0.4, 0.5]$	0.066978	0.06703	0.000069
$[0.5, 0.6]$	0.074085	0.074124	0.000052
$[0.6, 0.7]$	0.080563	0.080593	0.00004
$[0.7, 0.8]$	0.086554	0.086578	0.000032
$[0.8, 0.9]$	0.092156	0.092175	0.000025
$[0.9, 1.0]$	0.097434	0.097451	0.000023

intervals needing refinement (error > 10^{-4}): first 4 intervals

n)

implemented adaptive trapezoidal quadrature with tolerance $\mathrm{tol} = 10^{-5}$:

Function	Adaptive Result	Intervals Used	Uniform (part i)
e^x	1.7183	64	120
$\sqrt{x}$	0.6667	79	-

the adaptive algorithm automatically allocates computational effort where needed, making it more efficient than uniform refinement.

problem 2

a)

to transform the quadrature rule from $[-1, 1]$ to $(-3, 3)$ we use

$$x = \frac{b-a}{2}t + \frac{b+a}{2} = 3t$$

where $t \in [-1, 1]$, so $dx = 3dt$. then

$$\int_{-3}^{3} e^x dx = \int_{-1}^{1} e^{3t} \cdot 3dt = 3\int_{-1}^{1} e^{3t} dt$$

the gauß-legendre approximation is given by

$$G_h = 3\sum_{i=0}^{3} w_i e^{3x_i}$$

so

$$\begin{aligned} G_h &= 3\left(w_0 e^{3x_0} + w_1 e^{3x_1} + w_2 e^{3x_2} + w_3 e^{3x_3}\right) \\ &= 3(6.676229465096898) \\ &= 20.028688395290693 \end{aligned}$$

wheras the exact value is $e^3 - e^{-3} = 20.035749854819805$.

b)

from the given error formula, each subinterval of length $h = \frac{b-a}{m}$ contributes error

$$E_i \propto h^{2n+1} = \left(\frac{b-a}{m}\right)^{2n+1}$$

total error from m subintervals

$$E_m \propto m \cdot \left(\frac{b-a}{m}\right)^{2n+1} = \frac{(b-a)^{2n+1}}{m^{2n}}$$

we expect error to decrease when we subdivide the interval. more subintervals means more precision.

c)

let $f(x) = \frac{x^8}{8!}$

we can see that $f^{(2n)}(x) = \frac{x^{8-2n}}{(8-2n)!}$ for $n > 0$ and $\leqslant 8$ so

$$E_1 = \frac{6^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} \cdot \frac{\xi^{8-2n}}{(8-2n)!}$$

then sum the left and right errors to obtain

$$E_2 = \frac{3^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} \cdot \frac{\xi_1^{8-2n} + \xi_2^{8-2n}}{(8-2n)!}$$

which yields

$$\frac{E_2}{E_1} = \frac{\xi_1^{8-2n} + \xi_2^{8-2n}}{2^{2n+1}\xi^{8-2n}}$$

we can see that increasing n makes it more accurate very fast, as the numerator grows smaller exponentially and the denominator bigger.

problem 3

a)

to find an orthogonal basis $p_j(x)$ from the given canonical basis

$$\phi_0(x) \equiv 1,\ \phi_1(x) = x,\ \phi_2(x) = x^2,\ \phi_3(x) = x^3,$$

we can use the gram-schmidt process.

recall that

$$p_j(x) = \phi_j(x) - \sum_{k=0}^{j-1} \left\{ \frac{\int_0^1 \phi_j(x) p_k(x)\, dx}{\int_0^1 [p_k(x)]^2\, dx} \right\} p_k(x),$$

and notice that $p_0(x) \equiv 1$.

thus we obtain

$$p_1(x) = \phi_1(x) - \frac{\int_0^1 \phi_1(x)p_0(x)\,dx}{\int_0^1 [p_0(x)]^2\,dx} p_0(x)$$

$$= x - \frac{\int_0^1 x\,dx}{\int_0^1 dx} = x - \frac{1}{2},$$

$$p_2(x) = \phi_2(x) - \frac{\int_0^1 \phi_2(x)p_1(x)\,dx}{\int_0^1 [p_1(x)]^2\,dx} p_1(x)$$

$$- \frac{\int_0^1 \phi_2(x)p_0(x)\,dx}{\int_0^1 [p_0(x)]^2\,dx} p_0(x)$$

$$= x^2 - \frac{\int_0^1 x^2\left(x - \frac{1}{2}\right)dx}{\int_0^1 \left[x - \frac{1}{2}\right]^2 dx}\left(x - \frac{1}{2}\right)$$

$$- \frac{\int_0^1 x^2\,dx}{\int_0^1 dx} = x^2 - x + \frac{1}{6}$$

$$p_3(x) = \overset{\text{magic}}{\cdots} = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

b)

we perform polynomial division to find the remaining two roots and obtain

$$p_3(x) : \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{10}$$

$$\Rightarrow x = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3}{5}}$$

so

$$
\begin{aligned}
x_0 &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}, \\
x_1 &= \frac{1}{2}, \\
x_2 &= \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}.
\end{aligned}
$$

c)

let

$$w_i = \int_0^1 \ell_i(x)\,dx$$

with

$$\ell_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n-1} \frac{x - x_i}{x_k - x_i}$$

where $n - 1 = 2$ since $k = 0, 1, 2$.

computing the cardinal polynomials, we obtain

$$
\begin{aligned}
\ell_0(x) &= \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \\
&= \frac{x - \frac{1}{2}}{-\frac{1}{2}\sqrt{\frac{3}{5}}} \cdot \frac{x - \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}}{-\sqrt{\frac{3}{5}}} \\
&= \frac{5}{3}\left(-2x^2 + \sqrt{\frac{3}{5}}x + \frac{1}{2}\sqrt{\frac{3}{5}} - \frac{1}{2}\right) \\
\ell_1(x) &= \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} \\
&= \frac{x - \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}}{\frac{1}{2}\sqrt{\frac{3}{5}}} \cdot \frac{x - \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}}{\frac{1}{2}\sqrt{\frac{3}{5}}} \\
&= \frac{4}{3}\left(x^2 - x + \frac{1}{10}\right) \\
\ell_2(x) &= \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} \\
&= \frac{5}{3}\left(-2x^2 - \sqrt{\frac{3}{5}}x + \frac{1}{2}\sqrt{\frac{3}{5}} + \frac{1}{2}\right)
\end{aligned}
$$

integrating to find the weights

$$
\begin{aligned}
w_0 &= \int_0^1 \ell_0(x)\,dx = \frac{5}{18} \\
w_1 &= \int_0^1 \ell_1(x)\,dx = \frac{4}{9} \\
w_2 &= \int_0^1 \ell_2(x)\,dx = \frac{5}{18}
\end{aligned}
$$

by symmetry we have $w_0 = w_2$.

d)

the 3-point gauß-legendre quadrature rule for $[0, 1]$ is

$$\int_0^1 f(x)\,dx \approx \frac{5}{18}f(x_0) + \frac{4}{9}f\left(\frac{1}{2}\right) + \frac{5}{18}f(x_2)$$

where $x_0 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}$ and $x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}$.

to verify that the 3-point rule integrates x^5 exactly, we compute

$$\int_0^1 x^5\,dx = \frac{x^6}{6}\,\Big|_0^1 = \frac{1}{6}$$

and using the symmetry property and binomial expansion for $x_0^5 + x_2^5$ where $x_0 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}$ and $x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}$, we obtain

$$\frac{5}{18}\left(x_0^5 + x_2^5\right) + \frac{4}{9}\cdot\frac{1}{32} = \frac{1}{6}$$

∎

e)

to show equivalence with the standard table, we transform from the reference interval $[-1, 1]$ to $[0, 1]$ using:

$$x = \frac{t+1}{2} \quad \text{where } t \in [-1, 1] \text{ and } x \in [0, 1]$$

so

$$x_0 = \frac{-\frac{\sqrt{15}}{5} + 1}{2} = \frac{1}{2} - \frac{\sqrt{15}}{10}$$
$$x_1 = \frac{0 + 1}{2} = \frac{1}{2}$$
$$x_2 = \frac{\frac{\sqrt{15}}{5} + 1}{2} = \frac{1}{2} + \frac{\sqrt{15}}{10}$$

using the change of variables $x = \frac{t+1}{2}$, we have $dx = \frac{1}{2}\,dt$. so

$$w_0 = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$$
$$w_1 = \frac{8}{9} \cdot \frac{1}{2} = \frac{4}{9}$$
$$w_2 = \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$$

note that $\frac{\sqrt{15}}{10} = \frac{1}{2}\sqrt{\frac{3}{5}}$ so the transformed rule matches exactly our derived 3-point gauß-legendre quadrature for $[0, 1]$, confirming that both approaches yield the same integration rule.