exercise 2

these are my solutions to the second exercise set of TMA4135.

there should be a python source code file attached to this deliverable. this file can be used to generate the images shown in problem 2.

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problem 1

define the *cardinal functions* as

$$\ell_{i(x)} := \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$= \frac{x - x_0}{x_i - x_0} \cdot \ldots \cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \ldots \cdot \frac{x - x_n}{x_i - x_n},$$

for $i = 0, \ldots, n$.

a)

given $x_0 = -3, x_1 = -1, \text{and } x_2 = 5$, then

$$l_0 = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{x+1}{-2} \cdot \frac{x-5}{-8} = \frac{(x+1)(x-5)}{16} = \frac{x^2 - 4x - 5}{16}$$

$$l_1 = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_1 - x_j} = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{x+3}{2} \cdot \frac{x-5}{-6} = \frac{(x+3)(x-5)}{-12} = \frac{-x^2 + 2x + 15}{12}$$

$$l_2 = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_2 - x_j} = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x+3}{8} \cdot \frac{x+1}{6} = \frac{(x+3)(x+1)}{48} = \frac{x^2 + 4x + 3}{48}$$

b)

we can easily verify these cardinal functions

$$l_0(x_0) = \frac{(-3)^2 - 4(-3) - 5}{16} = \frac{9 + 12 - 5}{16} = 1$$
$$l_0(x_1) = \frac{(-1)^2 - 4(-1) - 5}{16} = \frac{1 + 4 - 5}{16} = 0$$
$$l_0(x_2) = \frac{5^2 - 4 \cdot 5 - 5}{16} = \frac{25 - 20 - 5}{16} = 0$$

$$l_1(x_0) = \frac{-(-3)^2 + 2(-3) + 15}{12} = \frac{-9 - 6 + 15}{12} = 0$$
$$l_1(x_1) = \frac{-(-1)^2 + 2(-1) + 15}{12} = \frac{-1 - 2 + 15}{12} = 1$$
$$l_1(x_2) = \frac{-5^2 + 2 \cdot 5 + 15}{12} = \frac{-25 + 10 + 15}{12} = 0$$

$$l_2(x_0) = \frac{(-3)^2 + 4(-3) + 3}{48} = \frac{9 - 12 + 3}{48} = 0$$
$$l_2(x_1) = \frac{(-1)^2 + 4(-1) + 3}{48} = \frac{1 - 4 + 3}{48} = 0$$
$$l_2(x_2) = \frac{5^2 + 4 \cdot 5 + 3}{48} = \frac{25 + 20 + 3}{48} = 1$$

thus we see that they follow the desired pattern.

c)

assume we have $n + 1$ data points, then

$$p_{n(x)} := \sum_{i=0}^{n} y_i \ell_{i(x)}$$

must be the (unique) interpolation polynomial for points (x_i, y_i).

to show this, we first must convince ourselves that $\deg(\ell_i(x)) = n$. recall the definition

$$\ell_{i(x)} := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$\Rightarrow \ell_{i(x)} = \prod_{\substack{j=0 \\ j \neq i}}^{n} f(x)$$

where

$$f(x) = \frac{x - x_j}{x_i - x_j}$$

$$\Rightarrow \deg(f(x)) = 1$$

the product contains exactly n factors, because we iterate over the $n + 1$ data points, minus the one for $i = j$. thus, $\deg\left(\ell_{i(x)}\right) = n$.

lastly, we must verify that the interpolation condition holds for $p_{n(x)}$

$$p_{n(x_i)} = y_i, \quad i = 0, ..., n.$$

we can expand the definition to show this

$$
\begin{aligned}
p_{n(x_i)} &= \sum_{j=0}^{n} y_j \ell_{j(x_i)} \\
&= \sum_{j=0}^{n} y_j \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} \\
&= \sum_{j=0}^{n} y_j \cdot g(i, j)
\end{aligned}
$$

where

$$
g(i, j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
$$

thus we get

$$
\begin{aligned}
p_{n(x_i)} &= \left(\sum_{\substack{j=0 \\ j \neq i}}^{n} y_j \cdot 0 \right) + \left(\sum_{\substack{j=0 \\ j = i}}^{n} y_j \cdot 1 \right) \\
&= (0) + (y_i) = y_i
\end{aligned}
$$

as such, we can see that $p_{n(x)}$ is the interpolation polynomial of the given data points.

d)

recall from a) that given $x_0 = -3, x_1 = -1, and\ x_2 = 5$, then

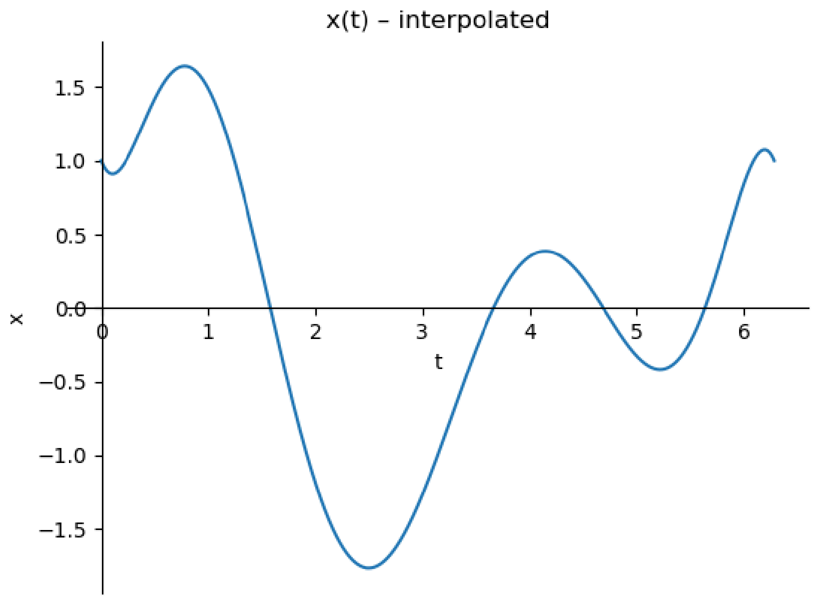
$$
l_0 = \frac{x^2 - 4x - 5}{16} \qquad l_1 = \frac{-x^2 + 2x + 15}{12} \qquad l_2 = \frac{x^2 + 4x + 3}{48}
$$

let $y_0 = 8, y_1 = -4, \text{and } y_2 = 12$. then using our proven formula from c), we can compute the interpolation polynomial

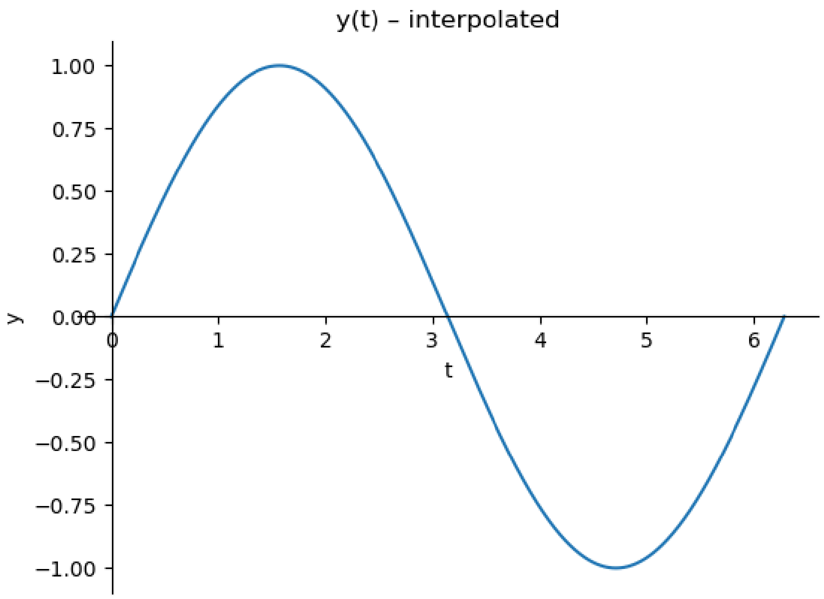
$$\begin{aligned}
p(x) &= \sum_{j=0}^{2} y_j \ell_{j(x)} \\
&= y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x) \\
&= \frac{x^2 - 4x - 5}{2} + \frac{x^2 - 2x - 15}{3} + \frac{x^2 + 4x + 3}{4} \\
&= \frac{6x^2 - 24x - 30 + 4x^2 - 8x - 60 + 3x^2 + 12x + 9}{12} \\
&= \frac{13x^2 - 20x - 81}{12}.
\end{aligned}$$

problem 2

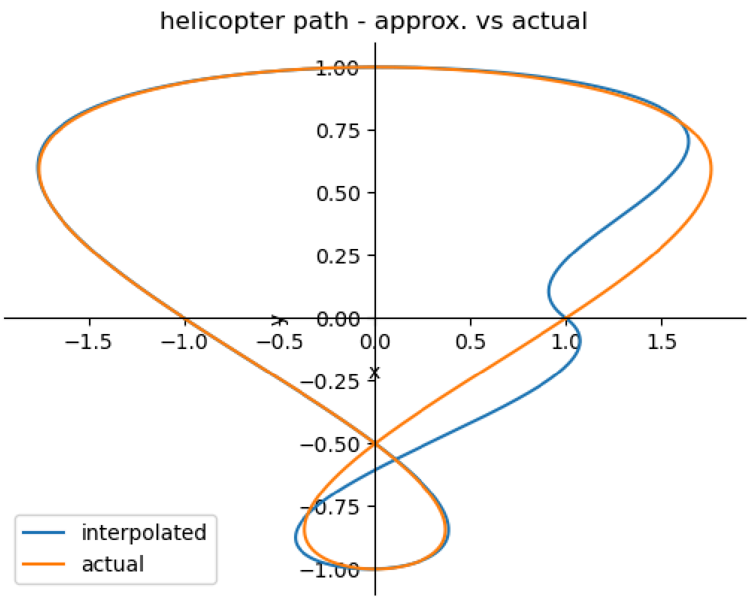
a)



b)


y(t) – interpolated
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
0
1
2
3
4
5
6
t

c) & d)


helicopter path - approx. vs actual
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x
interpolated
actual

problem 3

let $y = f(x) := 2 + \sin(x)\cos(x)$ with

i	0	1	2
x_i	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
y_i	2	$2+\frac{\sqrt{3}}{4}$	$2+\frac{\sqrt{3}}{4}$

and define

$$p(x) := a_1 + a_2 \sin x + a_3 \sin 2x$$

where $p(x_i) = y_i \text{ for } i \in \{0,1,2\} \text{ and } a_1, a_2, a_3 \in \mathbb{R}$.

a)

from some quick analysis, we can see that

$$a_1 = y_1 = 2 \Rightarrow$$
$$\frac{\sqrt{3}}{4} = \frac{a_2}{2} + \sqrt{3}\frac{a_3}{2} \wedge$$
$$\frac{\sqrt{3}}{4} = \sqrt{3}\frac{a_2}{2} + \sqrt{3}\frac{a_3}{2} \Rightarrow$$
$$\frac{a_2}{2} + \sqrt{3}\frac{a_3}{2} = \sqrt{3}\frac{a_2}{2} + \sqrt{3}\frac{a_3}{2}$$
$$a_2 = \sqrt{3}a_2 \Rightarrow a_2 = 0 \Rightarrow$$
$$a_3 = \frac{1}{2}$$

so we get $p(x) = 2 + \frac{1}{2}\sin 2x$.

b)

now, we can show that $p(x) = f(x)$, i.e. the error $e(x) = 0$

recall that

$$\sin 2x = 2 \sin x \cos x$$

we then get

$$\begin{aligned} p(x) &= 2 + \frac{1}{2} \sin 2x \\ &= 2 + \frac{1}{2}(2 \sin x \cos x) \\ &= 2 + \sin x \cos x = f(x) \end{aligned}$$

as expected.

problem 4

define

$$c_0 := y_0, \quad p_0(x) := c_0$$

and recursively

$$\begin{aligned} c_k &:= \frac{y_k - p_{k-1}(x_k)}{\prod_{i=0}^{k-1}(x_k - x_i)}, \\ p_k(x) &:= p_{k-1}(x) + c_k \prod_{i=0}^{k-1}(x - x_i), \end{aligned}$$

for $k = 1, ..., n$.

a)

given data points

i	0	1	2
x_i	-1	1	5
y_i	6	-2	4

compute $p_2(x)$.

first we need

$$\begin{aligned} p_1(x) &= p_0(x) + \frac{y_1 - p_0(x_1)}{x_1 - x_0} \cdot (x - x_0) \\ &= 6 + \frac{-2-6}{1-(-1)} \cdot (x - (-1)) \\ &= 6 - 4(x+1) = -4x + 2 \end{aligned}$$

then use $p_1(x)$ in

$$\begin{aligned} p_2(x) &= p_1(x) + \frac{y_2 - p_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} \cdot (x - x_0)(x - x_1) \\ &= -4x + 2 + \frac{4 - (-4) \cdot 4 + 2}{(5-(-1))(5-1)} \cdot (x - (-1))(x - 1) \\ &= -4x + 2 + \frac{11}{12} \cdot (x+1)(x-1) \\ &= \frac{11}{12}x^2 - 4x + \frac{13}{12} \end{aligned}$$

b)

$p_k(x)$ is the interpolation polynomial for the data points, because

$$\deg(p_k(x)) = k \quad \forall \quad k \in \mathbb{N} \tag{1}$$

and

$$p_k(x_i) = y_i \quad \forall \quad i = 0, ..., k \tag{2}$$

to show 1, we can see from the definition that the product has degree k, because there are k factors containing x. by inductive argument, we can trivially see that $p_{k-1}(x)$ must have a degree equal to $k-1$. as such,

$$\deg(p_k(x)) = k$$

next, (2) can be shown by a splitting the problem into two cases, one of them requiring induction, otherwise just direct proofs. observe

$$\begin{aligned} p_k(x_k) &= p_{k-1}(x_k) + c_k \prod_{i=0}^{k-1}(x_k - x_i) \\ &= \cancel{p_{k-1}(x_k)} + \frac{y_k - \cancel{p_{k-1}(x_k)}}{\cancel{\prod_{i=0}^{k-1}(x_k - x_i)}} \cdot \cancel{\prod_{i=0}^{k-1}(x_k - x_i)} \\ &= y_k. \end{aligned}$$

thus (2) holds for x_k. to show that it holds for x_j where $j < k$ requires that we analyze the product in the definition:

$$\prod_{i=0}^{k-1}(x_j - x_i)$$

but since $j \in \{0, ..., k-1\}$, the product must contain a 0-factor for any chosen j, thus canceling the entire second term of the definition, leaving us with

$$p_k(x_j) = p_{k-1}(x_j).$$

then we need to show that

$$p_{k-1}(x_j) = y_j \quad \forall \quad j \in \{0, ..., k-1\}$$

let us perform an inductive argument. the hypothesis is

$$p_0(x_j) = y_j$$

this is wrong.

c)

we can use the points from a) to form a scheme

$$\begin{array}{c|cccc}
x_0 = -1 & y_{0,0} = 6 & & \\
& & y_{0,1} = -4 & \\
x_1 = 1 & y_{1,1} = -2 & & y_{0,2} = 13/12 \\
& & y_{1,2} = 3/2 & \\
x_2 = 5 & y_{2,2} = 4 & &
\end{array}$$

then we can form the interpolation polynomial according to the formula

$$p_k(x) = \sum_{i=0}^{k} y_{0,i} \prod_{j=0}^{i-1} (x - x_j)$$

such that for $n + 1 = 3$ we get

$$
\begin{aligned}
p_n(x) = p_2(x) &= 6 - 4(x - (-1)) + \frac{13}{12}(x - (-1))(x - 1) \\
&= 6 - 4x - 4 + \frac{13}{12}\left(x^2 - 1\right) \\
&= \frac{13}{12}x^2 - 4x + \frac{11}{12}
\end{aligned}
$$

which is *almost* what we got in a) suggesting that i've made a slight error in either task.