exercise 9

these are my solutions to the ninth exercise set of TMA4135.

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problem 1

recall the definition of a periodic function f for a $p > 0$

$$f(x + p) = f(x) \quad \forall \quad x \in \mathbb{R}.$$

the smallest such p is called the fundamental period of f.

a)

"every periodic function has a fundamental period" is a false statement.

examine $f(x) = 1$, which is periodic since given a $p = 1$, then $f(x + 1) = f(x) = 1$ for all $x \in \mathbb{R}$. it is trivially periodic. however, there is no smallest p for which this holds:

given a $\delta > 0$ there is always a $\widehat{\delta} > 0$ for which $\widehat{\delta} < \delta$ holds. this is a property of the real number line, thus there is no fundamental period for this periodic function.

b)

1. "$\mathrm{Per}_p := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } p\text{-periodic}\}$ is a vector space" is a true statement since all periodic functions that are added together or scaled by some scalar are still periodic. this comes from the linear property of periodic functions.
   we can prove this by taking two periodic functions f and g and seeing if their linear combination is an element of the space

$$\begin{aligned} h(t) &:= a \cdot f(t) + b \cdot g(t) \\ &= a \cdot f(t + p_f) + b \cdot g\big(t + p_g\big) \\ &= a \cdot f\big(t + p_f \cdot p_g\big) + b \cdot g\big(t + p_g \cdot p_f\big) \\ &= h\big(t + p_f \cdot p_g\big) = h(t + p_h) \end{aligned}$$

thus the linear combination $h(t)$ must be periodic itself and therefore an element of Per_p. this proves scalar multiplication and vector addition axioms, rest is trivial.

2. "let $\phi : \mathbb{R} \to \mathbb{R} \Rightarrow \phi \circ f$ is p-periodic" is true, since it 'captures' the input of the function ϕ such that it becomes periodic itself, always reiterating over the same values.

$$
\begin{aligned}
(\phi \circ f)(t) &= \phi(f(t)) \\
&= \phi(f(t+p)) \\
&= (\varphi \circ f)(t+p)
\end{aligned}
$$

3. "let $n \in \mathbb{N}, a \in \mathbb{R} \Rightarrow f(x+a), f(nx), f(x/n)$ are p-periodic" is false, since scaling the input parameter changes the period.

$$
\begin{aligned}
&(1) \quad f(x+a) = f((x+p)+a) = f((x+a)+p)) \ \underline{\text{ok}} \\
&(2) \quad f(nx) = f(n(x+p)) = f(nx+np) = f(nx+p) \ \underline{\text{ok}} \\
&(3) \quad f(x/n) = f((x+p)/n) = f(x/n+p/n) \ \underline{\text{not ok}}
\end{aligned}
$$

(2) works because $n \in \mathbb{N}$, such that the period repeats n times. (3) doesn't work because dividing by a natural number causes the period to contract and thus isn't p-periodic anymore.

4. "the absolute difference between two periods p and p' is also a period of f" is true, since it just means that the periods occur periodically.

$$
\begin{aligned}
f(x) &= f(x+p') = f(x-p') \\
&= f((x+p)-p') = f(x+(p-p'))
\end{aligned}
$$

given that $p > p'$.

5. "let $a, b \in \mathbb{R} \Rightarrow \int_a^{a+p} f(x)\,dx = \int_b^{b+p} f(x)\,dx$" is true, since it says that integrating over the period is the same regardless of where you start integrating from.

   we can choose a $c \in \mathbb{R}$ such that

$$\begin{aligned}\int_a^{a+p} f(x)\,dx &= \int_a^{c} f(x)\,dx + \int_c^{a+p} f(x)\,dx \\ &= \int_c^{a+p} f(x)\,dx + \int_a^{c} f(x)\,dx = \int_b^{b+p} f(x)\,dx\end{aligned}$$

   because c is a midpoint chosen to displace the integral sum such that the startpoints of the integration are the same in terms of the period of the function f.

6. "if f is differentiable, f' is also p-periodic" is true

$$f(x) = f(x+p) \Longrightarrow f'(x) = f'(x+p)$$

c)

- $f(x) = \cos(2x+3)$ has a fundamental period $p = \pi$ since the $+3$ doesn't affect the period and then the usual period of 2π is halved by the coefficient in-front of x.
- $f(x) = \pi \sin\left(\frac{3}{2}\pi x\right)$ has a fundamental period $p = 4/3$ since the π in-front of the $\sin$-expression only affects the amplitude and $\frac{2\pi}{(3/2)\pi} = 4/3$.
- $f(x) = \cos\left(\frac{\pi}{m+1}x\right) + \sin\left(\frac{\pi}{n-1}x\right)$ for $m \in \mathbb{Z} \setminus \{-1\}, n \in \mathbb{Z} \setminus \{1\}$ can be broken into two functions that can be analyzed separately first.

- $g(x) := \cos\left(\frac{\pi}{m+1}x\right)$ has a fundamental period $p_g = 2(m+1)$.
- $h(x) := \sin\left(\frac{\pi}{n+1}x\right)$ has a fundamental period $f_h = 2(n+1)$.
- we can combine these two to obtain $f(x) = g(x) + h(x)$.

to find the fundamental period, we can draw inspiration from number theory to see that the combined period must be

$$\begin{aligned} \gcd(p_g, f_H) &= \gcd(2(m+1), 2(n+1)) \\ &= \gcd(m+1, n+1) \\ &\leqslant (m+1)(n+1) \\ &= mn + m + n + 1 \end{aligned}$$

but we cannot shorten this further using p-period algebra, so this must be the fundamental period for this wave.

problem 2

a)

$h(x) := f(x)g(x)$ is odd for odd f and even g, since

$$h(x) = -f(-x)g(-x) = -(f(-x)g(-x)) = -h(-x)$$

b)

if f and g are both odd or even, then $h(x) := f(x)g(x)$ is even

1. for the first case: f and g are both even

$$h(x) = f(x)g(x) = f(-x)g(-x) = h(-x)$$

2. for the second case: f and g are both odd

$$\begin{aligned} h(x) &= f(x)g(x) \\ &= (-f(-x))(-g(-x)) = f(-x)g(-x) = h(-x) \end{aligned}$$

c)

if f is odd and g is even then both $f \circ g$ and $g \circ f$ are even, since
1. $f(g(x)) = f(g(-x))$, since g is even
2. $g(f(x)) = g(-f(-x)) = g(f(-x))$

d)

if f is odd and $L > 0$,

$$\begin{aligned} \int_{-L}^{L} f(x)\,dx &= \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \\ &= \int_{0}^{L} f(-x)(-1)\,dx + \int_{0}^{L} f(x)\,dx \\ &= -\int_{0}^{L} f(-x)\,dx + \int_{0}^{L} f(x)\,dx \\ &= -\int_{0}^{L} f(x)\,dx + \int_{0}^{L} f(x)\,dx \\ &= 0 \end{aligned}$$

e)

if f is even and $L > 0$,

$$
\begin{aligned}
\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \\
&= \int_{0}^{L} f(-x)\,dx + \int_{0}^{L} f(x)\,dx \\
&= \int_{0}^{L} f(x)\,dx + \int_{0}^{L} f(x)\,dx \\
&= 2\int_{0}^{L} f(x)\,dx
\end{aligned}
$$

problem 3

$$a_0(f') = 0, \quad a_n(f') = nb_n(f), \quad b_{n(f')} = -na_{n(f)}$$

if $f(-\pi) \neq f(\pi)$ then the formulas don't work anymore, since there will be a discreet jump every time the period is restarted. to fix the formulas we need to account for this. let $[f] = f(\pi) - f(-\pi)$

$$
\begin{cases}
a_0(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{2\pi}[f(x)]_{-\pi}^{\pi} = \frac{[f]}{2\pi} \\
\\
a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx \\
\qquad = \frac{1}{\pi}\Big([f(x)\cos(nx)]_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\Big) \\
\qquad = \frac{1}{\pi}\big((-1)^n[f] + n\pi b_n(f)\big) \\
\qquad = nb_n(f) + \frac{1}{\pi}(-1)^n[f] \\
\\
b_n(f') = -na_n(f)
\end{cases}
$$

recall the fourier series for a p-periodic function f

$$f(x) \approx f_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi}{p} x \right) + b_n \sin\left(\frac{2\pi}{p} x \right) \right)$$

where

$$\begin{cases} a_0 = \frac{1}{p} \int_p f(x)\, dx \\ a_n = \frac{2}{p} \int_p f(x) \cos\left(\frac{2\pi}{p} n x \right) dx \\ b_n = \frac{2}{p} \int_p f(x) \sin\left(\frac{2\pi}{p} n x \right) dx \end{cases}$$

let $f(x) = \sin^2(x) + 3x^2 - 4x + 5$ be periodically continued based on $[-\pi, \pi]$.

notice that this series has a jump $[f] > 0$ so we need to use the modified formulas we found above.

also remark that $\sin^2(x) = \frac{1 - \cos(2x)}{2}$, which is already its fourier series.

since fourier series are linear, we can compute the fourier series of each term and we can use the provided properties to compute the coefficients using the derivative of f.

$$f'(x) = \cdots + 6x - 4$$

we then need to compute the fourier series for x^2, x and 1 using

$$\text{fourier}(1) = \begin{cases} a_0(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = 1 \\ a_n(1) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)\,dx = 0 \\ b_n(1) = 0 \end{cases}$$

$$\text{fourier}(x) = \begin{cases} a_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0 \\ a_n(x) = \frac{1}{n}(b_n(1)) = 0 \\ b_n(x) = \frac{1}{n}\left(a_n(1) - \frac{1}{\pi}(-1)^n[x]\right) = \frac{2}{n}(-1)^{n+1} \end{cases}$$

$$\text{fourier}\left(x^2\right) = \begin{cases} a_0\left(x^2\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{6\pi}\left[x^3\right]_{-\pi}^{\pi} = \frac{1}{3}\pi^2 \\ a_n\left(x^2\right) = -\frac{1}{n}b_n(2x) = -\frac{2}{n}b_n(x) = \frac{4}{n^2}(-1)^n \\ b_n\left(x^2\right) = \frac{1}{n}\left(a_n(2x) - \frac{1}{\pi}(-1)^n\left[x^2\right]\right) = 0 \end{cases}$$

combining these we get

$$\begin{aligned} \text{fourier}(f(x)) &= \text{fourier}\left(\sin^2(x)\right) \\ &+ \text{fourier}\left(3x^2\right) - \text{fourier}(4x) + \text{fourier}(5) \\ &= \frac{1}{2}(1 - \cos(2x)) + 3\ \text{fourier}\left(x^2\right) \\ &-4\ \text{fourier}(x) + 5\ \text{fourier}(1) \\ &= \frac{1}{2}(1 - \cos(2x))x + \begin{cases} a_0(f(x)) &= 3a_0\left(x^2\right) - 4a_0(x) + 5a_0(1) \\ &= \pi^2 + 5 \\ a_n(f(x)) = \frac{12}{n^2}(-1)^n \\ b_n(f(x)) = -\frac{8}{n}(-1)^{n+1} = \frac{8}{n}(-1)^n \end{cases} \\ &= \pi^2 + \frac{11}{2} - \frac{1}{2}\cos(2x) \\ &+ \sum_{n=1}^{\infty}\left(\frac{12}{n}\cos(nx) + 8\sin(nx)\right)\frac{(-1)^n}{n} \end{aligned}$$

problem 4

a)

to find the fourier series of

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \text{ or } \frac{\pi}{2} < x \leqslant \pi \\ x & \text{if } 0 \leqslant x \leqslant \frac{\pi}{2} \end{cases}$$

we can use the formulae for the coefficients

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} x\,dx \\ &= \frac{1}{2\pi}\left[x^2\right]_0^{\pi/2} \\ &= \frac{\pi}{8} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} x\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^{\pi/2} \\ &= \frac{1}{2n}\sin\!\left(\frac{\pi}{2}n\right) + \frac{1}{\pi n^2}\left(\cos\!\left(\frac{\pi}{2}n\right) - 1\right) \end{aligned}$$

then for the different cases of $n \bmod 4$

$$a_n = \begin{cases} 0 & \text{if } n \equiv 0 \\ \frac{1}{2n} - \frac{1}{\pi n^2} & \text{if } n \equiv 1 \\ -\frac{2}{\pi n^2} & \text{if } n \equiv 2 \\ -\frac{1}{2n} - \frac{1}{\pi n^2} & \text{if } n \equiv 3 \end{cases}$$

then for the $\sin$ terms

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} x\sin(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_0^{\pi/2} \\ &= \frac{1}{\pi n^2}\sin\left(\frac{\pi}{2}n\right) - \frac{1}{2n}\cos\left(\frac{\pi}{2}n\right) \end{aligned}$$

then similar $n \bmod 4$ computations

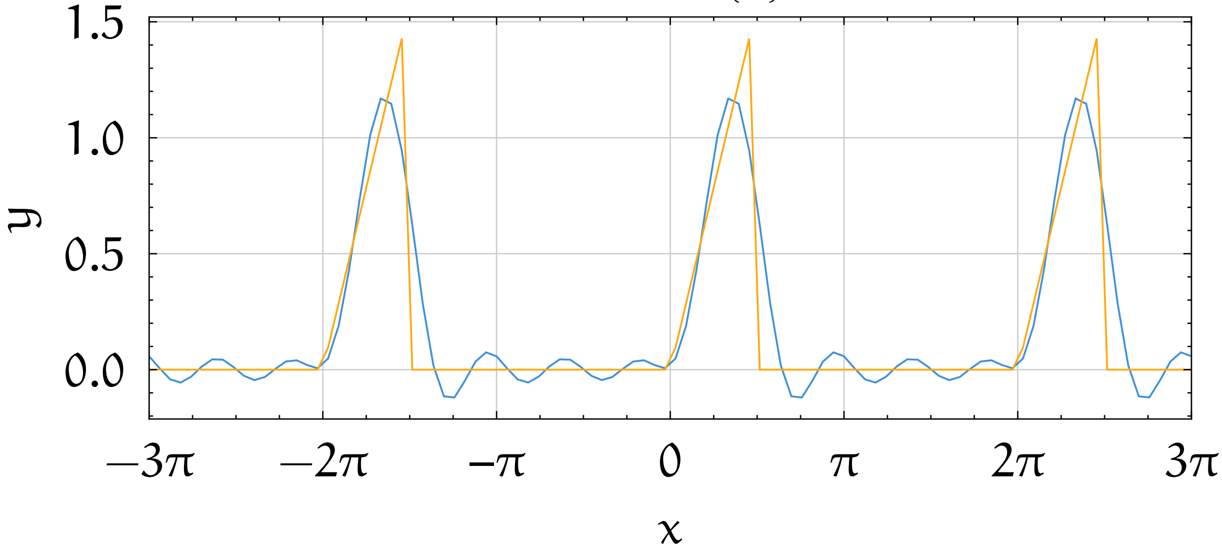
$$b_n = \begin{cases} -\frac{1}{2n} & \text{if } n \equiv 0 \\ \frac{1}{\pi n^2} & \text{if } n \equiv 1 \\ \frac{1}{2n} & \text{if } n \equiv 2 \\ -\frac{1}{\pi n^2} & \text{if } n \equiv 3 \end{cases}$$

then putting it all together

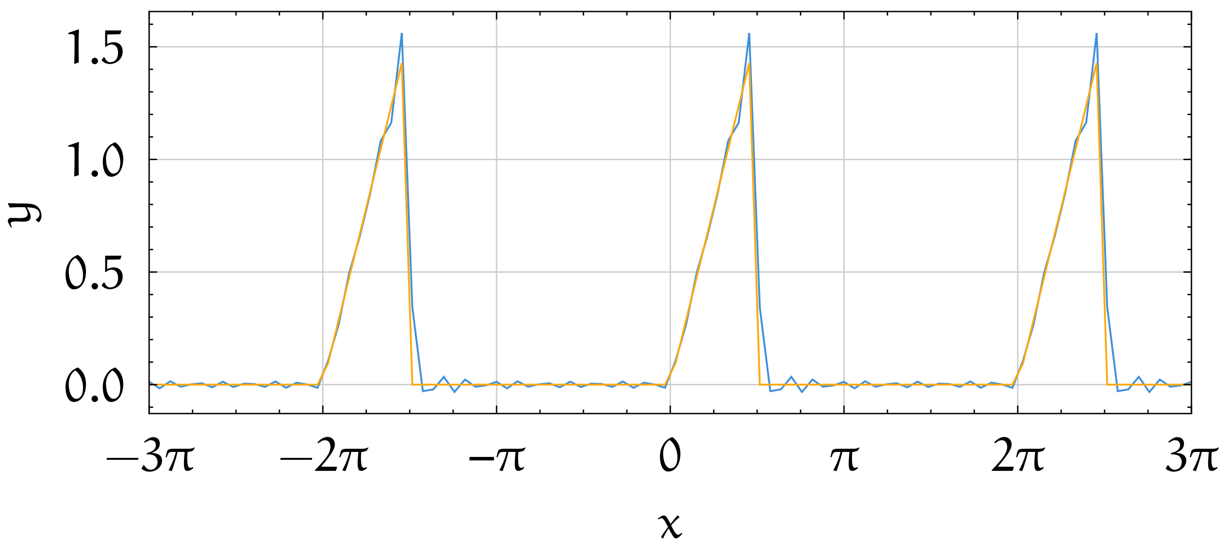
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

which is our final fourier series for $f(x)$.

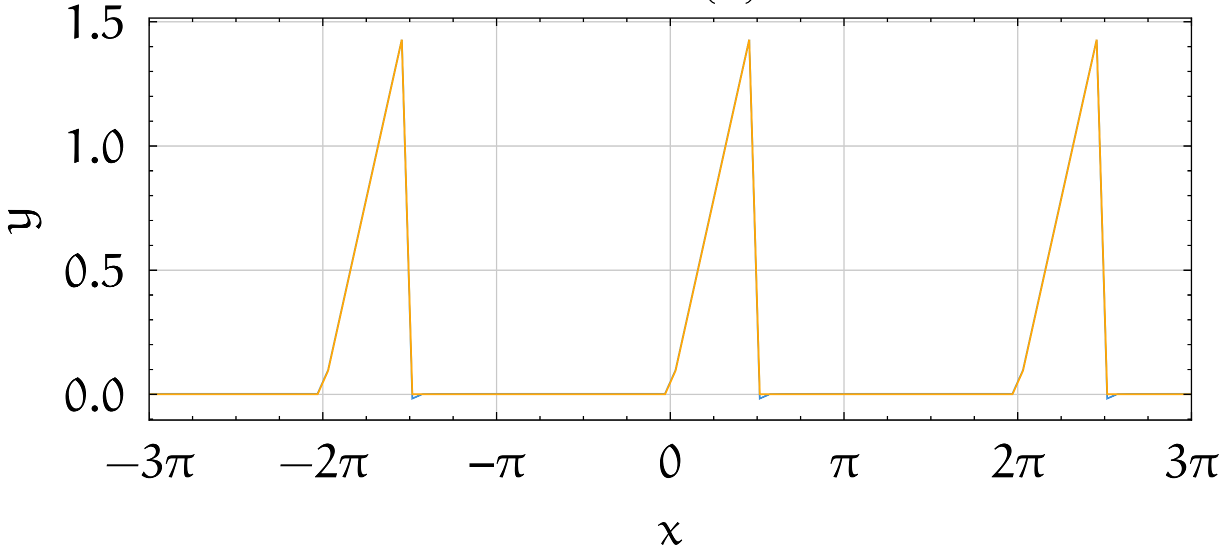
the fourier series of $f(x)$ with $N = 5$



the fourier series of $f(x)$ with $N = 20$



the fourier series of $f(x)$ with $N = 100$



b)

let

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \leqslant \pi \end{cases}$$

then we can proceed as usual to calculate the coefficients of the fourier series, utilising symmetries of x and $\pi - x$ on the given intervals.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ &= \frac{1}{\pi} \left(\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx \right) \\ &= \frac{2}{\pi} \int_0^{\pi/2} x\, dx \\ &= \frac{1}{\pi} \left[x^2 \right]_0^{\pi/2} \\ &= \frac{\pi}{4} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\left(\int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right) \\
&= \frac{1}{\pi}\left(\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^{\pi/2}\right. \\
&\left.+\left[\frac{\pi - x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right]_{\pi/2}^{\pi}\right) \\
&= \frac{1}{\pi}\left(\left[\cancel{\frac{\pi}{2n}\sin\left(\frac{\pi}{2}n\right)} + \frac{1}{n^2}\cos\left(\frac{\pi}{2}n\right) - \frac{1}{n^2}\right]\right. \\
&\left.+\left[\frac{1}{n^2}(-1)^{n+1} - \cancel{\frac{\pi}{2n}\sin\left(\frac{\pi}{2}n\right)} + \frac{1}{n^2}\cos\left(\frac{\pi}{2}n\right)\right]\right) \\
&= \frac{1}{\pi n^2}\left(2\cos\left(\frac{\pi}{2}n\right) - 1 + (-1)^{n+1}\right)
\end{aligned}
$$

notice that for odd n, $a_n = 0$, since the cosine is zero and the alternating sign is fixed to 1, canceling with the constant term. thus we can shorten it further with a definition for $n = 2k$

$$
\begin{aligned}
a_n &= \frac{1}{4\pi k^2}\left(2\cos(\pi k) - 1 + (-1)^{2k+1}\right) \\
&= \frac{1}{2\pi k^2}\left((-1)^k - 1\right) \\
&= \frac{2}{\pi n^2}\left((-1)^{n/2} - 1\right)
\end{aligned}
$$

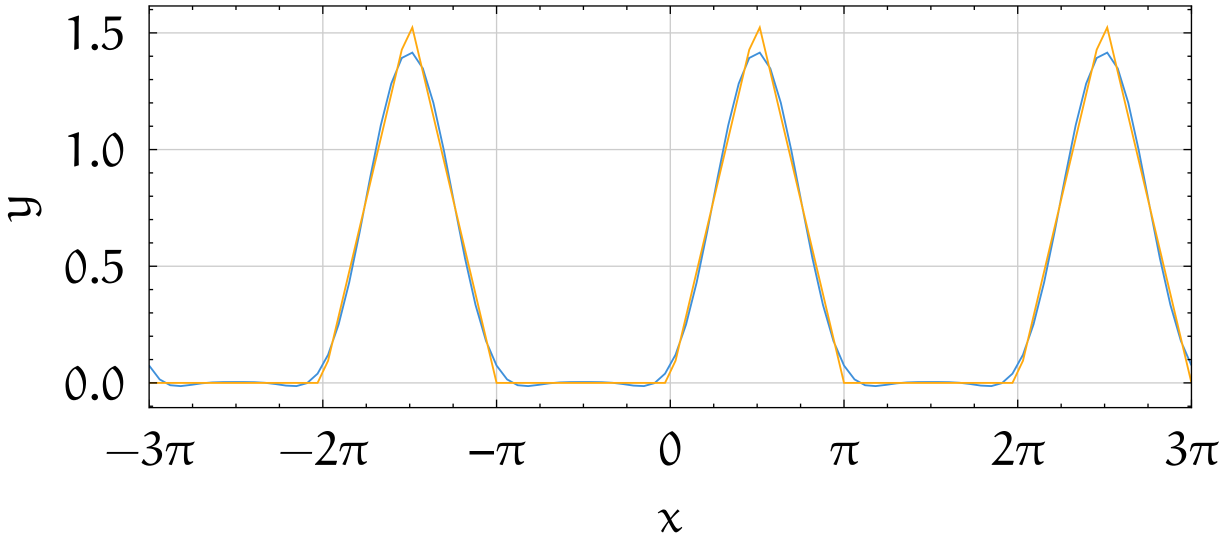
similarly b_n, we can recognize the symmetry earlier in our calculations

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\left(\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right) \\
&= \frac{1}{\pi}\big(1-(-1)^n\big)\int_0^{\pi/2} x\sin(nx)\,dx
\end{aligned}
$$

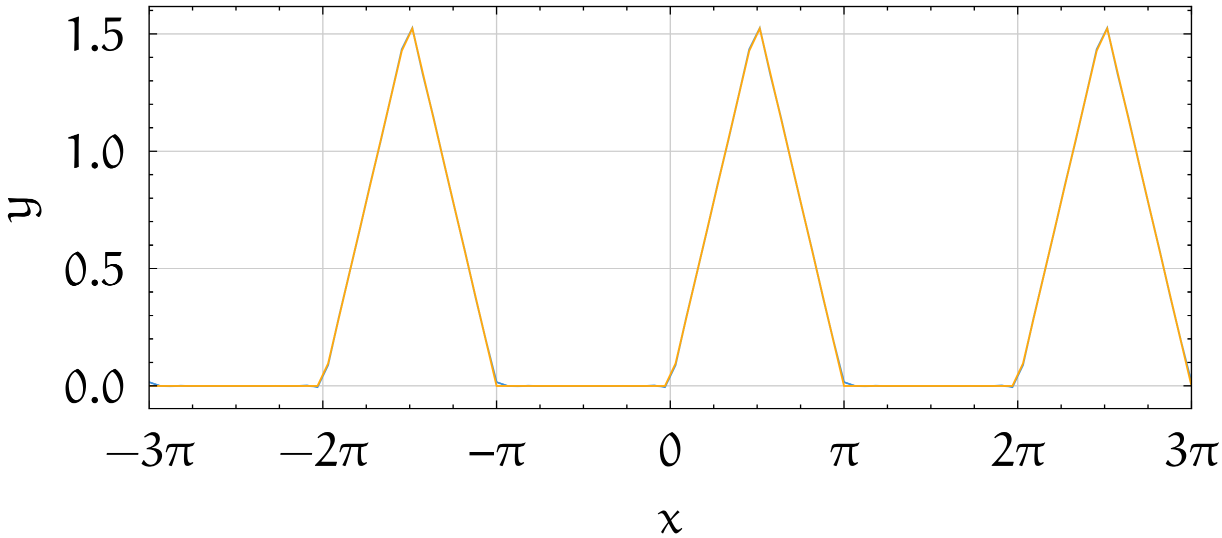
notice that for even n, $b_n = 0$, let $n = 2k+1$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\Big(1-(-1)^{2k+1}\Big)\int_0^{\pi/2} x\sin((2k+1)x)\,dx \\
&= \frac{1}{\pi}\Big(1+(-1)^{2k}\Big)\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_0^{\pi/2} \\
&= \frac{2}{\pi n^2}\sin\Big(\frac{\pi}{2}n\Big) \\
&= \frac{2}{\pi n^2}(-1)^{\frac{n-1}{2}}
\end{aligned}
$$

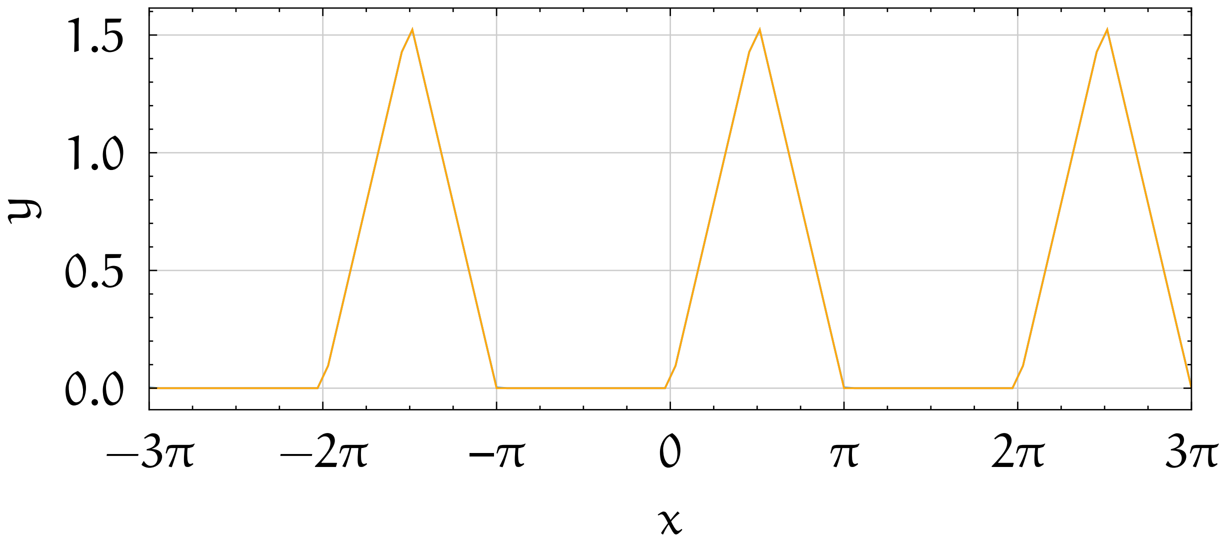
the fourier series of $f(x)$ with $N = 5$



the fourier series of $f(x)$ with $N = 20$



the fourier series of $f(x)$ with $N = 100$



c)

let

$$f(x) = \begin{cases} -\pi - x & \text{if } -\pi < x < -\frac{\pi}{2} \\ x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \leqslant \pi \end{cases}$$

then

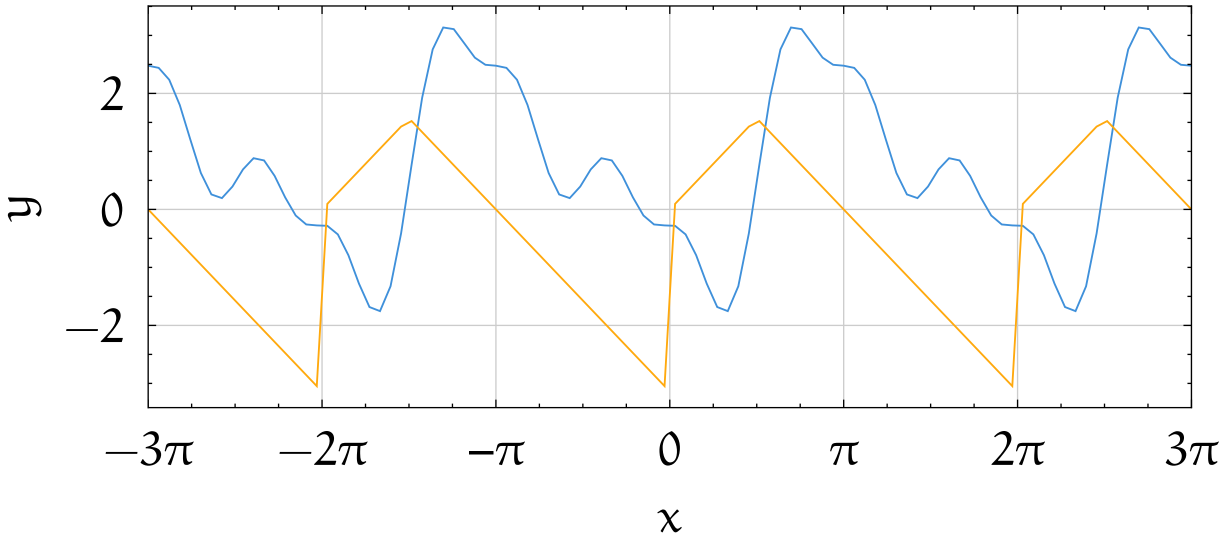
$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} (-\pi - x)\,dx + \cancel{\int_{-\pi/2}^{\pi/2} x\,dx} + \int_{\pi/2}^{\pi} (\pi - x)\,dx \right) \\ &= \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x)\,dx \\ &= \frac{2}{\pi}\left(\left[\pi^2 - \frac{\pi^2}{2}\right] - \frac{1}{2}\left[\pi^2 - \frac{\pi^2}{4}\right] \right) \\ &= 2\pi - \pi - \pi + \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\cos(nx)\,dx \\
&= \frac{2}{\pi}\left[\frac{1}{n^2}\cos(nx) + \frac{\pi - x}{n}\sin(nx)\right]_{\pi/2}^{\pi} \\
&= \frac{2}{\pi}\left(\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\cos\left(\frac{\pi}{2}n\right) - \frac{\pi}{2n}\sin\left(\frac{\pi}{2}n\right)\right)
\end{aligned}
$$

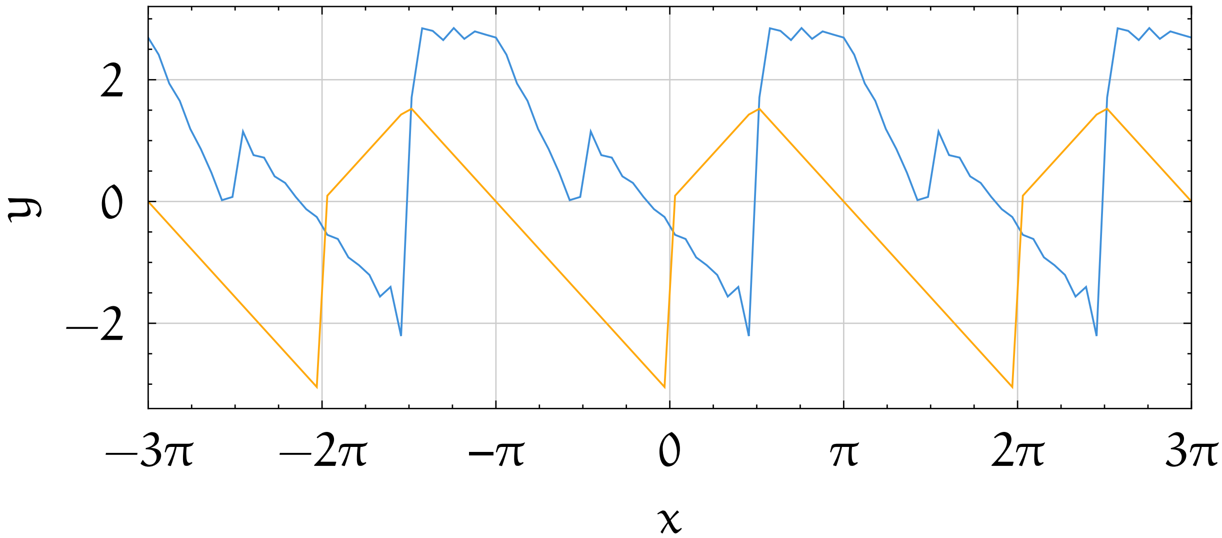
$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx + \int_{0}^{\pi/2} x\sin(nx)\,dx\right) \\
&= \frac{2}{\pi}\left(\left[\frac{x-\pi}{n}\cos(nx) - \frac{1}{n^2}\sin(nx)\right]_{\pi/2}^{\pi}\right. \\
&\left.+\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_{0}^{\pi/2}\right) \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n}\cos\left(\frac{\pi}{2}n\right) - \cancel{\frac{1}{n^2}\sin\left(\frac{\pi}{2}n\right)}\right. \\
&\left.+\cancel{\frac{1}{n^2}\sin\left(\frac{\pi}{2}n\right)} - \frac{\pi}{2n}\cos\left(\frac{\pi}{2}n\right)\right) \\
&= \frac{2}{n}\cos\left(-\frac{\pi}{2}n\right)
\end{aligned}
$$

we could figure out what these expressions equal in each case $(\mathrm{mod}\,4)$, but for the sake of brevity, i'll leave it at that.

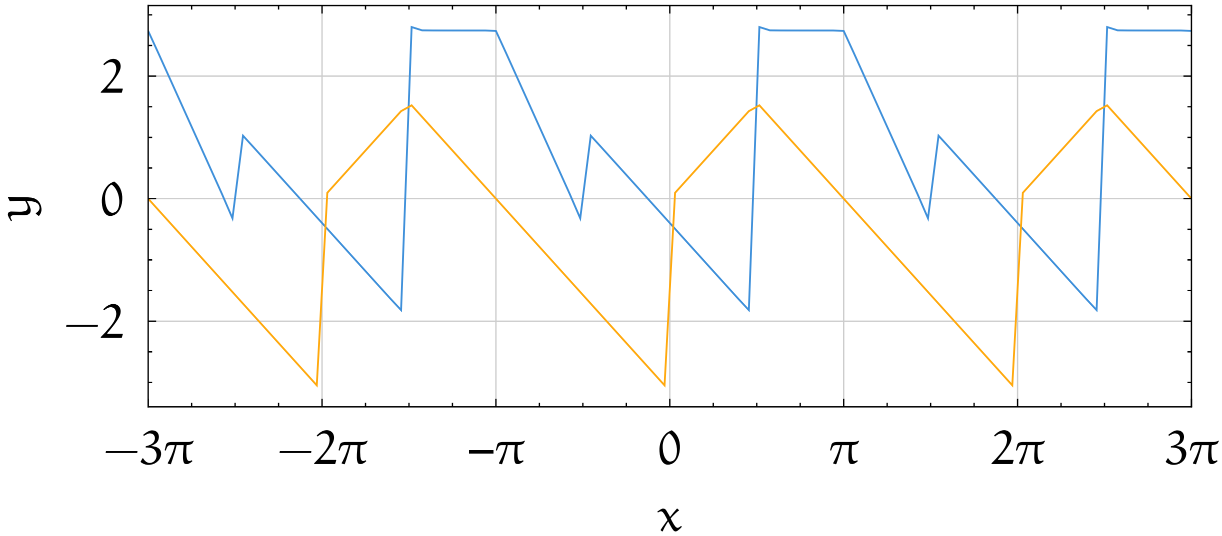
the fourier series of $f(x)$ with $N = 5$



the fourier series of $f(x)$ with $N = 20$



the fourier series of $f(x)$ with $N = 100$



as we can see, i’ve made a mistake in my calculations.