these are my solutions to the seventh exercise set of TMA4135.

this document was created using typst.

exercise 7

problem 1

a)

$$\begin{cases} 5y + 2y' + y'' + 3 = 0 \\ y(0) = 1 \\ y'(0) = 4 \end{cases}$$

we introduce the substitutions $y_1 = y$ and $y_2 = y_1' = y$ such that we get

$$5y_1 + 2y_2 + y_2{}' + 3 = 0$$
$$\Rightarrow y_2{}' = y'' = -3 - 2y_2 - 5y_1$$

for the first equation. this yields an autonomous first-order system

$$\begin{cases} y_1{}' = y_2 \\ y_2{}' = -3 - 2y_2 - 5y_1 \end{cases}$$

since there is no explicit t on the right sides of the equations, i.e. the state of the system only relies on its current state.

since we only use a step size of $h = 1$, we simply do

$$y_{n+1} = y_n + y_n',$$

so starting at $t = 0$ we get, using our initial values

$$\begin{cases} y_1(0) = 1 & \text{initial} \\ y_2(0) = 4 & \text{initial} \\ y_1'(0) = y_2(0) = 4 \\ y_2'(0) = -3 - 2y_2(0) - 5y_1(0) = -16 \end{cases}$$

which we can plug in to get our next values for y_1 and y_2

$$\begin{cases} y_1(1) = y_1(0) + y_1'(0) = 5 \\ y_2(1) = -12 \end{cases}$$

which concludes the first step in euler's method.

b)

for $k = 0, 1, 2, 3$

$$\begin{cases} y + 3y'' + 4y^{(3)} + \sin(t) = 3 \\ y^{(k)}(0) \qquad\qquad\qquad\quad = 4 - k \end{cases}$$

yields five equations.

in this case, we can make yet more substitutions

$$\begin{cases} y_0 \; = y \\ y_k' = y_{k+1} \; = y^{(k+1)} \end{cases}$$

such that

$$\begin{cases} y_0 + 3y_2 + 4y_3 + \sin(t) = 3 \\ y_k(0) = 4 - k \end{cases}$$

$$\Rightarrow \begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -\frac{1}{4}(y_1 + 3y_3 + \cos(t)) \end{cases}$$

so starting at $t = 0$ with step size $h = 1$ we obtain

$$y_0(0) = 4, \quad y_1(0) = 3, \quad y_2(0) = 2, \quad y_3(0) = 1$$

and

$$y_0'(0) = 3, \quad y_1'(0) = 2, \quad y_2'(0) = 1, \quad y_3'(0) = -7/4$$

such that after one step we have

$$\begin{cases} y_0(1) = y_0(0) + y_0'(1) = 7 \\ y_1(1) = 5 \\ y_2(1) = 3 \\ y_3(1) = -3/4 \end{cases}$$

c)

we have the system of ODEs

$$\begin{cases} 2u + \cos(u)\sin(u') + 3(u'')^3 + \ln(t+1) & = 42 \\ v^2 - 2vv' + (v'')^2 + v^{(3)} & = t \\ u^{(k)}(0) & = k^2, \quad k = 0, 1 \\ v^{(k)}(1) & = 1, \quad k = 0, 1, 2 \end{cases}$$

this time we have two functions of t, u and v. we can substitute as usual, then each one independently, since the equations don't overlap.

let

$$u^{(k)} = u_k \quad \text{and} \quad v^{(k)} = v_k$$

such that we obtain the two disjoint systems

$$\begin{cases} 2u_0 + \cos(u_0)\sin(u_1) + 3(u_2)^3 + \ln(t+1) & = 42 \\ u_k(0) & = k^2, \quad k = 0, 1 \end{cases}$$

$$\wedge \begin{cases} v_0^2 - 2v_0v_1 + v_2^2 + v_3 & = t \\ v_k(1) & = 1, \quad k = 0, 1, 2 \end{cases}$$

which yields

$$\begin{cases} u_0' = u_1 \\ u_1' = u_2 \\ u_2' = \frac{d}{dt}\sqrt[3]{\frac{1}{3}(42 - 2u_0 - \cos(u_0)\sin(u_1) - \ln(t+1))} \end{cases}$$

$$\wedge \begin{cases} v_0' = v_1 \\ v_1' = v_2 \\ v_2' = v_3 \\ v_3' = \frac{d}{dt}\left(t - v_0^2 + 2v_0v_1 - v_2^2\right) \end{cases}$$

combined with the initial conditions

$$\begin{cases} u_0(0) = 0 \\ u_1(0) = 1 \end{cases} \wedge \begin{cases} v_0(1) = 1 \\ v_1(1) = 1 \\ v_2(1) = 1 \end{cases}$$

yields for step size $h = 1$

$$\begin{cases} u_0(1) = u_0(0) + u_1(0) = 1 \\ u_1(1) = u_1(0) + u_1'(0) \\ \qquad = 1 + \sqrt[3]{\frac{1}{3}(42 - 2u_0(0) - \cos(u_0(0))\sin(u_1(0)) - \ln(0+1))} \\ \qquad = 1 + \sqrt[3]{\frac{42 - \sin(1)}{3}} \approx 3.3939 \end{cases}$$

and

$$\begin{cases} v_0(2) = v_0(1) + v_1(1) = 2 \\ v_1(2) = v_1(1) + v_2(1) = 2 \\ v_2(2) = v_2(1) + v_2'(1) \\ \qquad\; = 1 + \left(1 - (v_0(1))^2 + 2v_0(1)v_1(1) - (v_2(1))^2\right) \\ \qquad\; = 2 \end{cases}$$

thus have I done a step of euler's method for the system of ODEs.

problem 2

recall that a function $f : D \to V$ is lipschitz continuous iff

$$\forall y_1, y_2 \in D, \quad \underbrace{|f(y_2) - f(y_1)| \leqslant L|y_2 - y_1|}_{\text{lipschitz condition}}$$

for some $L \in \mathbb{R}$.

a)

let $f : \mathbb{R}^2 \to \mathbb{R}, \quad (t, y) \mapsto \frac{y}{t^2}$.

converting the function to it's polar coordinate form using

$$t = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta)$$

such that $f[r, \theta] = \frac{\sin(\theta)}{r\cos^2(\theta)} = \frac{1}{r}\sec(\theta)\tan(\theta)$

the magnitude of the gradient of the function tells us how much the function changes in a point

$$
\begin{aligned}
\boldsymbol{\nabla} f[r,\theta] &= \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}\right)^{\mathsf{T}} \\
&= \left(-\frac{1}{r^2}\,\mathit{sec}(\theta)\,\mathit{tan}(\theta), \frac{1}{r}\,\mathit{sec}(\theta)\big(\mathit{sec}^2(\theta) + \mathit{tan}^2(\theta)\big)\right)^{\mathsf{T}} \\
\implies |\boldsymbol{\nabla} f[r,\theta]| &= \sqrt{\frac{1}{r^4}\,\mathit{sec}^2(\theta)\,\mathit{tan}^2(\theta) + \frac{1}{r^2}\,\mathit{sec}^2(\theta)\big(\mathit{sec}^2(\theta) + \mathit{tan}^2(\theta)\big)^2} \\
&= \frac{1}{r^2}\,\mathit{sec}(\theta)\sqrt{\mathit{tan}^2(\theta) + r^2\big(\mathit{sec}^2(\theta) + \mathit{tan}^2(\theta)\big)^2}
\end{aligned}
$$

if we look at $\mathit{lim}_{r\to\infty}|\boldsymbol{\nabla} f[r,\theta]|$ we observe that the function is well-behaved – i.e. lipschitz continuous – for values of θ that do not yield $\mathit{tan}(\theta) = \pm\infty$, since it converges to 0 due to $1/r^2$.

we need to inspect the expression more closely for those other values of θ. let $\varphi = \pi/2 - \theta$ so we can observe

$$
\begin{aligned}
\mathit{sec}(\theta) &= \mathit{sec}\Big(\frac{\pi}{2} - \varphi\Big) = \mathit{csc}(\varphi) \approx \frac{1}{\varphi} \\
\mathit{tan}(\theta) &= \mathit{tan}\Big(\frac{\pi}{2} - \varphi\Big) = \mathit{cot}(\varphi) \approx \frac{1}{\varphi}
\end{aligned}
$$

such that

$$
\begin{aligned}
\lim_{\varphi\to 0^+} |\boldsymbol{\nabla} f[r,\varphi]| &= \lim_{\varphi\to 0^+} \frac{1}{r^2\varphi}\sqrt{\frac{1}{\varphi^2} + r^2\left(\frac{1}{\varphi^2} + \frac{1}{\varphi^2}\right)^2} \\
&= \lim_{\varphi\to 0^+} \frac{1}{r^2\varphi}\sqrt{\frac{1}{\varphi^2} + r^2\frac{4}{\varphi^4}} \\
&= \lim_{\varphi\to 0^+} \frac{1}{r^2\varphi}\sqrt{4\frac{r^2}{\varphi^4}\left(1 + \frac{\varphi^2}{4r^2}\right)} \\
&= \lim_{\varphi\to 0^+} \frac{2}{r\varphi^3}\sqrt{1 + \frac{\varphi^2}{4r^2}} \\
&= \underbrace{\lim_{\varphi\to 0^+} \frac{2}{r\varphi^3}}_{\infty} \cdot \underbrace{\lim_{\varphi\to 0^+} \sqrt{1 + \frac{\varphi^2}{4r^2}}}_{\sqrt{1}=1} \\
&\to \infty
\end{aligned}
$$

we can see that the derivative diverges for such problematic angles. this shows that it is not lipschitz continuous for all its domain, since its rate of change becomes unbounded for certain input values. ∎

it was not really worth it going via polar coordinates.

b)

let $f(t,y) = \frac{\sin(t)}{t}y$ on $\mathbb{R}$.

the only point that may be problematic here is $f(0,0)$.

recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, such that

$$
\lim_{(t,y)\to(0,0)} f(t,y) = \lim_{(t,y)\to(0,0)} \frac{\sin(t)}{t}y = \lim_{(t,y)\to(0,0)} y = 0
$$

we need to check the rate of change in this point, so let

$$f_t(0+\varepsilon) = \frac{\sin(t)}{t}\varepsilon$$

but

$$\frac{\sin(t)}{t}\varepsilon \leqslant L\varepsilon$$

so the rate of change is bounded in y-direction. similarly for t-direction, we observe that the function is well-behaved. it is thus lipschitz continuous on $\mathbb{R}$.

problem 4

a)

given the explicit midpoint method

$$y_{n+1} = y_n + hf(t_n + 0.5h, y_n + 0.5hf(t_n, y_n)),$$

notice $k_1 = f(t_n, y_n)$ appears at depth 1 meaning the funcion is only nested twice, such that our number of stages $s = 2$. furthermore, we have $c_2 = 0.5$ from the time step and $a_{21} = 0.5$ from the function step. we can also observe that $b_1 = 1$ and $b_2 = 0$. this can be visualized in a mnemonic butcher tableau.

$$\begin{array}{c|cc} 0 & & \\ 0.5 & 0.5 & \\ \hline & 1 & 0 \end{array}$$

b)

I don't wanna write python right now, so I'm gonna solve this in typst. yes, I'm coding this in the typesetting language I use to write this text.

the value for y at $t = 1$ is 1.2699

```
let h = 0.01;
let f(t, y) = calc.exp(-y * y);
let y_1(t_0, y_0) = y_0
    + h * f(t_0 + 0.5 * h,
            y_0 + 0.5 * h * f(t_0, y_0));
let t = 0;
let y = 1;
while t < 1 {
  y = y_1(t, y);
  t += h;
};
[the value for $y$ at $t = 1$ is
#calc.round(y, digits: 4)]
```

problem 8

a)

we are given a system of ODEs describing the movement of a pendulum

$$\begin{cases} \theta''(t) = -\frac{g}{L} \sin(\theta(t)) \\ \theta(0) = \frac{\pi}{4} \\ \theta'(0) = 0 \end{cases}$$

where $g = 9.81 \mathrm{~m} \mathrm{~s}^{-2}$ and $L = 1 \mathrm{~m}$.

let $\theta_k = \theta^{(k)}$ in

$$\begin{cases} \theta_0(0) = \frac{\pi}{4} \\ \theta_1(0) = 0 \\ \theta_2(t) = -\frac{g}{L}\sin(\theta_0(t)) \\ \theta_0' = \theta_1 \\ \theta_1' = \theta_2 \end{cases}$$

b)

recall euler's method: $y_{n+1} = y_n + hy_n'$

θ is at $t = 1$ approximately -0.7775 (with step size $h = 0.01$ so i can reuse the code later).

```
let h = 0.01; let g = 9.81; let L = 1;

let t = 0;
let theta_0 = calc.pi / 4;
let theta_1 = 0;
let data = ();
while t < 5 {
  data.push((t, theta_0, theta_1));
  theta_0 += h * theta_1;
  theta_1 += h * (-g/L * calc.sin(theta_0));
  t += h;
};
data
```

c)

recall heun's method:

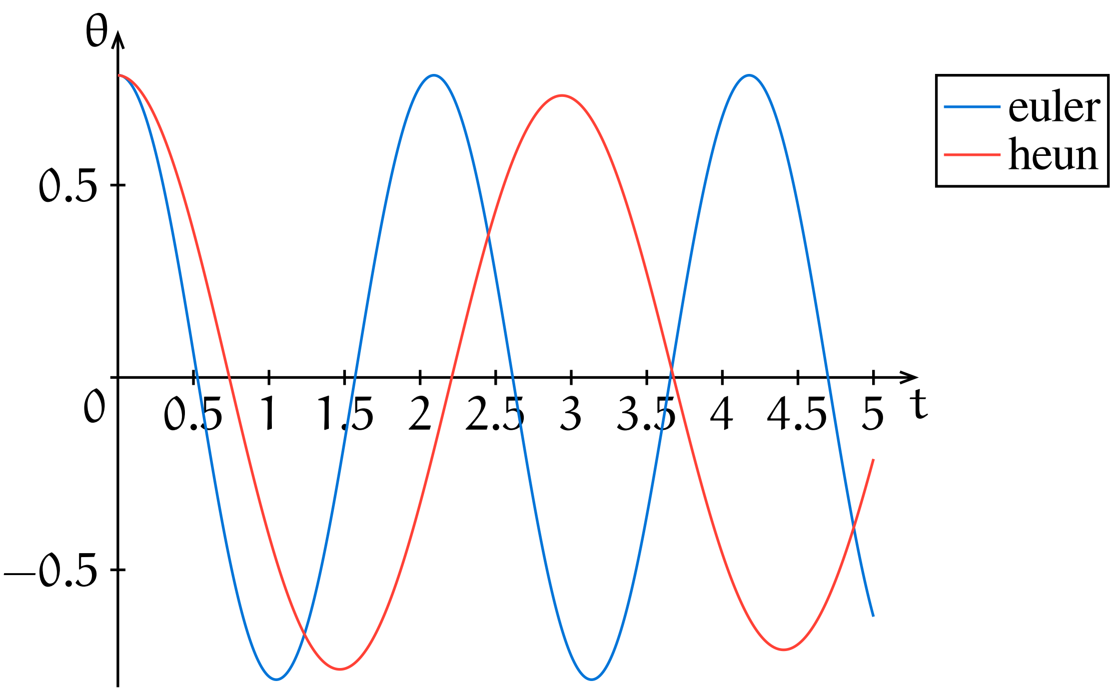
$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

θ is approxmately equal to -0.4111 at $t = 1$ (with step size $h = 0.01$, so i can reuse the code later).

```
let h = 0.01; let g = 9.81; let L = 1;

let t = 0;
let theta_0 = calc.pi / 4;
let theta_1 = 0;
let data = ();
while t < 5 {
  data.push((t, theta_0, theta_1));
  let euler_theta_1 = h * (-g/L
          * calc.sin(theta_0 + h * theta_1));
  theta_0 += h/2 * (theta_1 + euler_theta_1);
  theta_1 += h * (-g/L * calc.sin(theta_0));
  t += h;
};
data
```

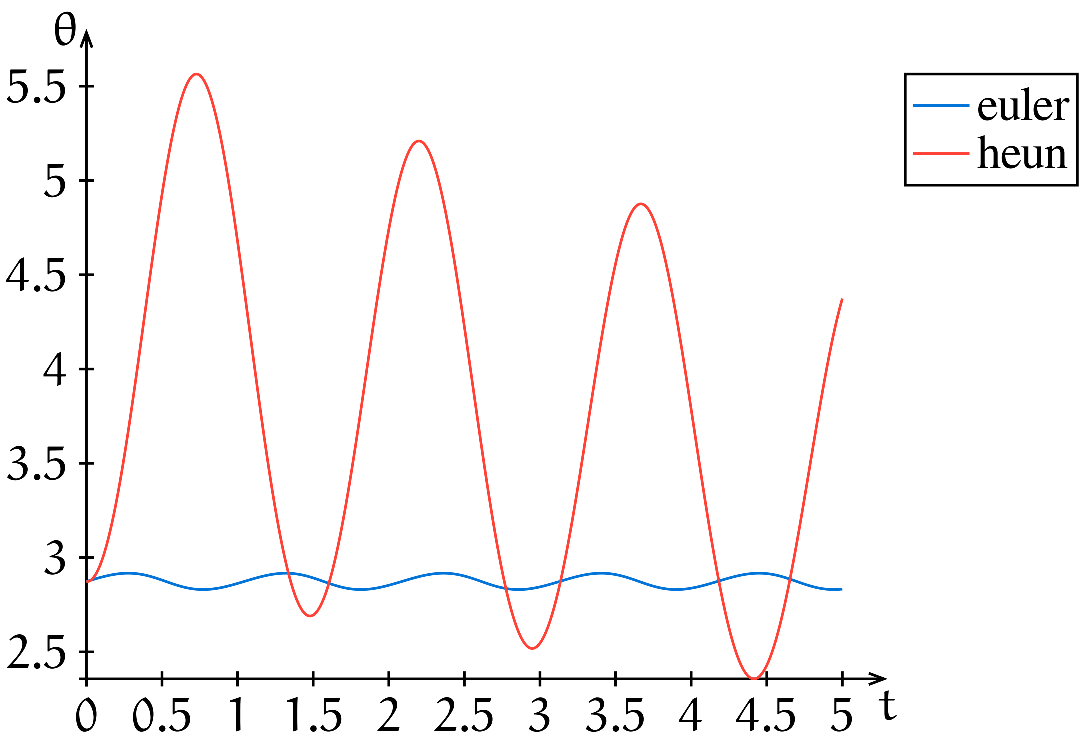
d)



I'm thought heun's method was wrongly implemented, because I didn't bother using explicit function definitions, which complicates things, but it seems to be more correct than euler's method.

e)

using the function $E(\theta_0, \theta_1) = \frac{1}{2}L^2(\theta_1)^2 + gL(1 - \cos(\theta_0))$ I can calculate the energy levels for both graphs at each point in time.



we can see that neither are constant. this is bad (probably). however, euler's method seems to be most correct in this sense too, if we are modelling a system devoid of friction, the energy should remain the same.

problem 10

consider

$$\begin{cases} y' = -2ty^2 \\ y(0) = 1 \end{cases}$$

a)

we can solve for the exact solution by integrating both sides

$$\frac{dy}{dt} = -2ty^2$$
$$\frac{1}{y^2}\,dy = -2t\,dt$$
$$\int y^{-2}\,dy = \int -2t\,dt$$
$$-\frac{1}{y} = -t^2 + C$$
$$y = \frac{1}{t^2 + C}$$

and since $y(0) = 1$ we obtain

$$y(0) = \frac{1}{0^2 + C} = \frac{1}{C} = 1 \Rightarrow C = 1$$

thus

$$y(t) = \frac{1}{t^2 + 1}.$$

plugging in $t = 0.4$ yields

$$
\begin{aligned}
y(0.4) = y\left(\frac{2}{5}\right) &= \frac{1}{\left(\frac{2}{5}\right)^2 + 1} \\
&= \frac{1}{\frac{4}{25} + 1} \\
&= \frac{25}{4 + 25} \\
&= \frac{25}{29} \\
&= 0.86206896551724
\end{aligned}
$$

as expected.

b)

recall that euler's method is $y_{n+1} = y_n + hf(t_n, y_n)$, and let $f(t, y) = -2ty^2$.

then, with a step size of $h = 0.1$ we get

$$
\begin{aligned}
y(0) &= 1, \\
y(0.1) &= 1 + 0.1 \cdot \left(-2 \cdot 0 \cdot 1^2\right) = 1, \\
y(0.2) &= 1 + 0.1 \cdot \left(-2 \cdot 0.1 \cdot 1^2\right) = 1 - 0.02 = 0.98, \\
y(0.3) &= 0.98 + 0.1 \cdot \left(-2 \cdot 0.2 \cdot 0.98^2\right) = 0.941584, \\
y(0.4) &= 0.888389
\end{aligned}
$$

after four steps of euler's method.

the error is

$$
e_4 := |y_4 - y(0.4)| = 0.02632.
$$

c)

recall heun's method is

$$\tilde{y}_{n+1} = y_n + hf(t_n, y_n) \quad \text{euler's method}$$
$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})]$$

thus with $h = 0.2$, two steps of heun's method looks like

$$\begin{aligned} y(0) &= 1 \\ \tilde{y}(0.2) &= 1 + 0.2 \cdot \left(-2 \cdot 0 \cdot 1^2\right) = 1 \\ y(0.2) &= 1 + 0.1\left[0 - 2 \cdot 0.2 \cdot 1^2\right] = 0.99 \\ \tilde{y}(0.4) &= 0.99 + 0.2 \cdot \left(-2 \cdot 0.2 \cdot 0.99^2\right) = 0.911592 \\ y(0.4) &= 0.911592 + 0.1 \cdot \left(-0.39204 - 2 \cdot 0.4 \cdot 0.911592^2\right) \\ &= 0.805908 \end{aligned}$$

this yields an error of

$$e_2 := 0.056161$$

which is greater than for euler. curious.

d)

$$\begin{aligned} k_1 &= f(t_n, y_n) = f(0, 1) = 0 \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) = -2 \cdot 0.2 \cdot 1^2 = -0.1 \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) = -2 \cdot 0.2 \cdot 0.98^2 = -0.38416 \\ k_4 &= f(t_n + h, y_n + hk_3) = -2 \cdot 0.4 \cdot (1 - 0.4 \cdot 0.38416) = -0.6770688 \end{aligned}$$

thus we obtain

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= 1 + \frac{0.4}{6}(0 + 2(-0.1) + 2(-0.38416) - 0.6770688) \\
&= 0.890307
\end{aligned}
$$

which yields the error

$$e_1 := 0.028238$$

which is also worse than euler, suprpsingly, but impressive for only a single step.

we can conclude that euler performed best, but it was given a much finer granularity, requiring four explicit steps to get to something slightly better than that which rk4 achieved in a single step.

I think I understand at least some numerics and ODEs now… 😅