TMA 4135 - Exercise 5

Problem 1

a) if f and g are two piecewise continuous functions of exponential order such that

$$\mathcal{L}\{f(t)\} = F(s) \quad \text{and}$$
$$\mathcal{L}\{g(t)\} = G(s),$$

then $\mathcal{L}(f \cdot g) \neq \mathcal{L}(f) \cdot \mathcal{L}(g)$, since

$$\mathcal{L}(f \cdot g) = \int_0^\infty e^{-st} f(t) g(t) \, dt$$
$$\neq \left(\int_0^\infty e^{-st} f(t) \, dt\right) \cdot \left(\int_0^\infty e^{-st} g(t) \, dt\right)$$
$$= \mathcal{L}(f) \cdot \mathcal{L}(g) \qquad \square$$

b) if f is a piecewise continuous function of exponential order and let $k \in \mathbb{N}$, then

$$\mathcal{L}\left\{\frac{d^k}{dt^k} t^k f(t)\right\} = s^k \mathcal{L}\{t^k f(t)\},$$

because let $g(t) = t^k f(t)$ and recall that

$$\mathcal{L}\left\{\frac{d^n}{dt^n} g(t)\right\} = s^n \mathcal{L}\{g(t)\},$$

but this is what we wanted $\square$

c) let $\mathcal{L}\{f(t)\}$ and $a \in \mathbb{R}$,

recall that

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}.$$

We want to arrive at this.

Recall also the shifting theorem:

for $\mathcal{L}(f) = F$,

$$\Rightarrow \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Thus notice $\mathcal{L}(f)(s-a) = \mathcal{L}\{e^{at} f(t)\}$

and similarly $\mathcal{L}(f)(s+a) = \mathcal{L}\{e^{-at} f(t)\}$.

So we obtain

$$\cancel{\mathcal{L}}\{\cosh(at) f(t)\} = \cancel{\mathcal{L}}\overbrace{\left\{\frac{f(t)\left(e^{at} + e^{-at}\right)}{2}\right\}}^{\text{linearity}}$$

$$\Leftrightarrow \quad \cosh(at) f(t) = f(t) \cdot \underbrace{\frac{e^{at} + e^{-at}}{2}}_{\cosh(at)}$$

Thus the statement holds.

Problem 2

1) Let $f(t) = \sinh(t) \cos(t)$

$$\Rightarrow \mathcal{L}(f) = \int_0^\infty e^{st} \cdot \sinh t \cdot \cos t \, dt$$

$$= \int_0^\infty e^{st} \cdot \left(\frac{e^t - e^{-t}}{2} \right) \cdot \cos t \, dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{st+t} - e^{st-t} \right) \cdot \cos t \, dt$$

$$= \frac{1}{2} \left[\mathcal{L}\{e^t \cos t\} - \mathcal{L}\{e^{-t} \cos t\} \right]$$

$\Updownarrow$ freq. shift thm. $\Updownarrow$

$$= \frac{1}{2} \left[\mathcal{L}\{\cos(t-1)\} - \mathcal{L}\{\cos(t+1)\} \right]$$

Now recall that

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\Rightarrow \frac{1}{2} \left[\frac{s-1}{(s-1)^2 + 1} - \frac{s+1}{(s+1)^2 + 1} \right]$$

$$= \frac{1}{2} \left[\frac{s-1}{s^2 - 2s + 2} - \frac{s+1}{s^2 + 2s + 2} \right]$$

$$= \frac{1}{2} \cdot \frac{(s-1)(s^2+2s+2) - (s+1)(s^2+2s+2)}{(s^2 - 2s + 2)(s^2 + 2s + 2)}$$

$$= \frac{1}{2} \cdot \frac{\cancel{s^3} \cancel{- s^2} \cancel{+ 2s^2} \cancel{- 2s} \cancel{+ 2s} - 2 \cancel{- s^3} \cancel{- s^2} - 2s^2 - 2s - 2s - 2}{s^4 \cancel{+ 2s^3} \cancel{+ 2s^2} \cancel{- 2s^3} \cancel{- 4s^2} \cancel{- 4s} \cancel{+ 2s^2} \cancel{+ 4s} + 4}$$

$$= -\frac{1}{2} \cdot \frac{2s^2+4s+4}{s^4+4}$$

$$= -\frac{s^2+2s+2}{s^4+4} = \mathcal{L}\{f(t)\}$$

note: s^2+2s+2
$= (s+1)^2+1$

2) Let $f(t) = 3t^2e^{-3t} + 7t^5 - 5t^7$

$$\Rightarrow \mathcal{L}(f) = \mathcal{L}\{3t^2e^{-3t} + 7t^5 - 5t^7\}$$

$$= 3\mathcal{L}\{t^2e^{-3t}\} + 7\mathcal{L}\{t^5\} - 5\mathcal{L}\{t^7\}$$

Use shifting on t^2e^{-3t}

$$\mathcal{L}\{t^2e^{-3t}\} = \mathcal{L}\{t^2\}(s+3).$$

And recall

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow 3\mathcal{L}\{t^2\}(s+3) + 7\mathcal{L}\{t^5\}(s)$$
$$-5\mathcal{L}\{t^7\}(s)$$

$$= 3 \cdot \frac{2}{(s+3)^3} + 7 \cdot \frac{120}{s^6} - 5 \cdot \frac{120 \cdot 6 \cdot 7}{s^8}$$

$$= \frac{6}{s^3+9s^2+27s+27} + \frac{840}{s^6} - \frac{25200}{s^8}$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

1 1
1 2 1
1 3 3 1

$$(840s^2 - 25200)(s^3+9s^2+27s+27)$$

$$= 840(s^5+9s^4+27s^3+27s^2)$$
$$\quad - 25200(s^3+9s^2+27s+27)$$

$$= 840s^5 + 7560s^4 + 22680s^3 + 22680s^2$$
$$\quad - 25200s^3 - 226800s^2 - 680400s$$
$$\quad - 680400$$

$$= 840s^5 + 7560s^4 - 2520s^3 - 204120s^2$$
$$\quad - 680400s - 680400$$

$$\Rightarrow (6s^8 + 840s^5 + 7560s^4 - 2520s^3$$
$$\quad - 204120s^2 - 680400s - 680400)$$
$$\div (s^{11} + 9s^{10} + 27s^9 + 27s^8) = \mathcal{L}(f)$$

Maybe I shouldn't have combined the fractions...

3) Let $f(t) = e^t \dfrac{d^{2024}}{dt^{2024}}\left(e^{-t} t^{2025}\right)$

$$= -e^t \cdot e^{-t} (\sim)$$
$$= -(t^{2025} + 2025t^{2024} + \dots)$$
$$= -\sum_{i=0}^{2025} \frac{2025!}{(2025-i)!} t^{2025-i}$$

D	I
t^{2025}	e^{-t}
$2025t^{2024}$	$-e^{-t}$
$\vdots$	$\vdots$
$2025!$	

$$= -2025! \sum_{n=0}^{2025} \frac{t^n}{n!}$$ This is the 2025th e^t - taylor polynomial

$$= -2025! \left(e^t - \sum_{n=2026}^{\infty} t^n/n! \right)$$ interesting, but unuseful

Recall again that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}(f) = -2025!\, \mathcal{L}\left\{ \sum_{n=0}^{2025} \frac{t^n}{n!} \right\}$$

$$= -2025! \sum_{n=0}^{2025} \mathcal{L}\left\{\frac{t^n}{n!}\right\}$$

$$= -2025! \sum_{n=0}^{2025} \frac{1}{s^{n+1}}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$ for $r \neq 1$

$$= \frac{-2025!}{s} \left(\frac{1-(1/s)^{2026}}{1-1/s} \right)$$

$$= -2025! \left(\frac{s^{2026}-1}{s^{2026}(s-1)} \right) = \mathcal{L}(f)$$

4) Let $f(t) = e^{-2t}\cos^2(3t) - 3t^2 e^{3t}$

$$\Rightarrow \mathcal{L}(f) = \mathcal{L}\{e^{-2t}\cos^2(3t)\}(s) - 3\,\mathcal{L}\{t^2 e^{3t}\}(s)$$

$$= \mathcal{L}\{\cos^2(3t)\}(s+2) - 3\,\mathcal{L}\{t^2\}(s-3).$$

$$2\cos^2(3t) = \cos(6t) + 1$$

$$\rightarrow \underbrace{\mathcal{L}\{\cos(6t)\}(s+2)}_{(s+2)/((s+2)^2+6^2)} + \underbrace{\mathcal{L}\{1\}(s+2)}_{1}$$

$$-3\underbrace{\mathcal{L}\{t^2\}(s-3)}_{2/(s-3)^3}$$

$$= \frac{s+2}{s^2+4s+40} + 1 - \frac{6}{s^3-9s^2+27s-27}$$

Let's leave it at that this time...

5) let $f(t) = \sqrt{t}$

Idea: use $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ with

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n \, dt$$

$$\sqrt{t} = t^{1/2} \xrightarrow{\mathcal{L}} \frac{1/2!}{s^{3/2}}$$

$$= s^{-3/2} \cdot \Gamma(3/2)$$

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$$

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} \, dt$$

$$= \int_0^\infty e^{-u^2} \cdot \frac{1}{u} \, 2u \, du \quad \Big| \; u(t) = \sqrt{t}$$

$$= 2\int_0^\infty e^{-u^2} \, du = \underbrace{\int_{-\infty}^\infty e^{-u^2} \, du}_{\text{Gaussian } \int \cdots dt}$$

$$= \sqrt{\pi}$$

$$\Rightarrow \mathcal{L}(f) = s^{-3/2} \cdot \frac{\sqrt{\pi}}{2}$$

Problem 3

a) Let $F(s) = \dfrac{3s+4}{s^2+4s+5} = \dfrac{3s+4}{(s+2)^2+1}$.

We want to shift the frequencies by -2 so let $s' = s+2$

$$\Rightarrow F(s') = \frac{3(s'-2)+4}{s'^2+1} = \frac{3s'-2}{s'^2+1}$$

$$= 3 \cdot \frac{s'}{s'^2+1} - 2 \cdot \frac{1}{s'^2+1},$$

but recall that

$$\mathcal{L}\{\cos(\omega t)\}(s) = \frac{s}{s^2+\omega^2},$$

$$\mathcal{L}\{\sin(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2}.$$

So

$$\mathcal{L}^{-1}\{F(s')\} = 3\mathcal{L}^{-1}\left\{\frac{s'}{s'^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s'^2+1}\right\}$$

$$= 3\cos t' - 2\sin t',$$

But we need to apply the shift

$$\Rightarrow \mathcal{L}^{-1}(F) = e^{-2t}(3\cos t - 2\sin t).$$

b) Let $F(s) = \frac{s}{(s^2+s+2)(s+4)}$

Using the Heaviside cover-up method we can split this fraction and perform the usual shifting algebra to find an f,

$$\mathcal{L}(f) = F(s)$$

To start, note that $s^2+s+2=0$

has no real solution. Form the terms:

$$F(s) = \frac{As+B}{s^2+s+2} + \frac{C}{s+4}$$

$$\Rightarrow C = \frac{-4}{16-4+2} = \underline{-\frac{2}{7}},$$

$$\rightarrow s = (As+B)(s+4) - \frac{2}{7}(s^2+s+2),$$

let $s=0 \rightarrow 0 = 4B - \frac{4}{7}$

$$\Rightarrow \underline{B = \frac{1}{7}},$$

$$\rightarrow s = \left(As+\frac{1}{7}\right)(s+4) - \frac{2}{7}(s^2+s+2),$$

let $s=1 \rightarrow 1 = 5A + \frac{5}{7} - \frac{8}{7}$

$$\Rightarrow \underline{A = \frac{2}{7}}$$

$$\Rightarrow F(s) = \frac{2s+1}{7(s^2+s+2)} - \frac{2}{7(s+4)}$$

We have now reduced the problem to finding

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+s+2}\right\} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}.$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$ can be seen as a shifted constant, i.e. let

$$\underline{g(t) = e^{-4t}} \xrightarrow{\mathcal{L}} \int_0^\infty e^{-st} e^{-4t}\, dt$$

$$= \int_0^\infty e^{-st-4t}\, dt = \int_0^\infty e^{-(s+4)t}\, dt = \frac{1}{s+4}$$

- $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+s+2}\right\}$ is likely a trigonometric expression. Notice that

$$s^2+s+2 = \left(s+\tfrac{1}{2}\right)^2 + \tfrac{7}{4}$$

Yes, this is a bad start...

So we can shift by $1/2$

$$\frac{2(s+1/2)+1}{(s+1/2)^2+(s+1/2)+2} = \frac{2s+2}{s^2+s+1/4+s+1/2+2}$$

$$= \frac{2s+2}{s^2+2s+1+\frac{7}{4}} = 2\cdot\frac{s+1}{(s+1)^2+\frac{7}{4}}$$

Shifting by $1/2$ yields a cosine-expression that is shifted by -1, so

$$\mathcal{L}^{-1}\{\cdots\} = \underline{2\cdot e^{-1/2 t}\cdot\cos\left(\frac{\sqrt{7}}{2}t\right)}$$

$1/2 - 1 = -1/2$

Thus we collect the terms

$$\mathcal{L}^{-1}(F) = \frac{2}{7}\left[e^{-1/2t}\cos\left(\frac{\sqrt{7}}{2}t\right) - e^{-4t}\right].$$

c) Let $F(s) = \dfrac{as}{s^2-2as+a^2+1}$, $a \in \mathbb{R}$

$$\Rightarrow F(s) = \frac{as}{(s-a)^2+1} = a\left(\frac{s'}{s'^2+1} + \frac{a}{s'^2+1}\right),$$

where $s' = s-a$ is our shift

$$\Rightarrow \mathcal{L}^{-1}(F) = ae^{at}(\cos t + \sin t)$$

Problem 4

a) $2y'' + y' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$

<u>Idea:</u> Take the laplace transformation of both sides and use differentiation of laplace transformations to turn the ODE into a regular equation. Then invert the transformation again.

$$2\mathcal{L}(y'') + \mathcal{L}(y') - \mathcal{L}(y) \quad \mathcal{L}(y) = Y(s)$$

$$\begin{tikzcd} y(t) \arrow[r, dashed, "y=f"] \arrow[d, "\mathcal{L}"'] & f(t) \\ Y(s) \arrow[r, "Y=F"'] & F(s) \arrow[u, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

$$= 2\left(s^2 Y(s) - s\cancel{y'(0)} - \cancel{y(0)}\right)$$

$$+ sY(s) - \cancel{y(0)} - Y(s)$$

$$= 2\left(s^2 Y(s) - s\right) + sY(s) - Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{2s}{2s^2 + s - 1} = \frac{s}{s^2 + \frac{1}{2}s - \frac{1}{2}}$$

$$= \frac{s}{(s + 1/4)^2 - 7/16}, \quad \text{let } s' = s + 1/4$$

$$\Rightarrow \frac{s'}{s'^2 - 7/16} - \frac{1/4}{s'^2 - 7/16} \qquad \frac{1}{4} \cdot \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}/4}{s'^2 - (\sqrt{7}/4)^2}$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} \; \underline{\underline{y(t) = e^{-1/4}\left[\cosh\left(\frac{\sqrt{7}}{4}t\right) - \frac{\sqrt{7}}{7}\sinh\left(\frac{\sqrt{7}}{4}t\right)\right]}}$$

b) $y''(t) + y(t) = 1, \quad y(0) = 0, \quad y'(0) = 0$

$$\Rightarrow \mathcal{L}(y'') + \mathcal{L}(y) = s^2 Y(s) - \cancel{s\,y'(0)} - \cancel{y(0)}$$

$$+ Y(s) = Y(s)\left(s^2+1\right) = \mathcal{L}(1) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)\,s} = \frac{As+B}{s^2+1} + \frac{C}{s}$$

$$\Rightarrow C = 1, \quad 1 = As^2 + Bs + C(s^2+1)$$

$$\Rightarrow A = -1, \quad B = 0$$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] & \begin{array}{c} 1 \\ s \\ s^2 \end{array} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow Y(s) = \frac{-s}{s^2+1} + \frac{1}{s}$$

$$\Rightarrow \underline{\underline{y(t) = 1 - \cos t}}$$

c) $2y'' + 10y' + 12y = 0, \quad y(0) = -2, \quad y'(0) = 1$

$$\Rightarrow 2\left(s^2 Y - s\,y'(0) - y(0)\right) + 10\left(sY - y(0)\right) + 12Y$$

$$= 2s^2Y - 2s + 2 + 10sY + 2 + 12Y$$

$$= 2s^2Y + 10sY + 12Y - 2s + 4 = 0$$

$$\Rightarrow Y(s) = \frac{2s-4}{2s^2+10s+12} = 2\cdot\frac{s-2}{s^2+5s+6} = 2\,\frac{s-2}{(s+2)(s+3)},$$

$$s-2 = A(s+3) + B(s+2) \Rightarrow A = 5, \quad B = -4$$

$$\Rightarrow Y(s) = 2\cdot\left(\frac{5}{s+2} - \frac{4}{s+3}\right)$$

$$\Rightarrow \underline{\underline{y(t) = 2\left[5e^{-2t} - 4e^{-3t}\right]}}$$

Sadly no wee-wuh...