exercise 1

these are my solutions to the first exercise set of TMA4135.

i recommend using a PDF-reader with document rotation capabilities, like zathura.

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problem 1

a)

$u(x,y,t)$	u_y	u_t	u_{xx}	u_{xy}	u_{yx}
$t^4 - \cos(xy)$	$x\sin(xy)$	$4t^3$	$y^2\cos(xy)$	$xy\cos(xy) + \sin(xy)$	$xy\cos(xy) + \sin(xy)$
$-\sin(txy)$	$-tx\cos(txy)$	$-xy\cos(txy)$	$t^2y^2\sin(txy)$	$t^2xy\sin(txy)$	$t^2xy\cos(txy)$
$e^{-t}\sin(x)\ln(y)$	$\left(e^{-t}\sin(x)\right)/y$	$-e^{-t}\sin(x)\ln(y)$	$-e^{-t}\sin(x)\ln(y)$	$\left(e^{-t}\cos(x)\right)/y$	$\left(e^{-t}\cos(x)\right)/y$
$e^{-x}\sqrt{x^3+y^2}$	$(2ye^{-x})/\sqrt{x^3+y^2}$	0	(†)	(‡)	(††)
$\left(te^t\right)\sin(x)$	0	$e^t(t+1)\sin(x)$	$-te^t\sin(x)$	0	0
$\sin(t)e^{-x} + \cos(t)e^{-y}$	$-\cos(t)e^{-y}$	$\cos(t)e^{-x} - \sin(t)e^{-y}$	$\sin(t)e^{-x}$	0	0

some calculations

(†)

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2} e^{-x}\sqrt{x^3+y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{3x^2e^{-x}}{2\sqrt{x^3+y^2}} - e^{-x}\sqrt{x^3+y^2}\right) \\
&= \frac{3}{2}\cdot\frac{\left(2xe^{-x} - x^2e^{-x}\right)\sqrt{x^3+y^2} - \frac{3x^4e^{-x}}{2\sqrt{x^3+y^2}}}{x^3+y^2} - \frac{3x^2e^{-x}}{2\sqrt{x^3+y^2}} + e^{-x}\sqrt{x^3+y^2} \\
&= \frac{3e^{-x}\left(\left(2x-x^2\right)\left(x^3+y^2\right) - 3x^4\right)}{4\left(x^3+y^2\right)^{\frac{3}{2}}} - \frac{6x^2e^{-x}\left(x^3+y^2\right)}{4\left(x^3+y^2\right)^{\frac{3}{2}}} + \frac{4e^{-x}\left(x^3+y^2\right)^2}{4\left(x^3+y^2\right)^{\frac{3}{2}}} \\
&= \frac{3e^{-x}\left(2xy^2 - x^5 - x^2y^2 - x^4\right) - 6x^2e^{-x}\left(x^3+y^2\right) + 4e^{-x}\left(x^6 + 2x^3y^2 + y^4\right)}{4\left(x^3+y^2\right)^{\frac{3}{2}}} \\
&= \frac{6xy^2e^{-x} - 3x^5e^{-x} - 3x^2y^2e^{-x} - 3x^4e^{-x} - 6x^5e^{-x} + 6y^2e^{-x} + 4x^6e^{-x} + 8x^3y^2e^{-x} + 8y^4e^{-x}}{4\left(x^3+y^2\right)^{\frac{3}{2}}} \\
&= \frac{6xy^2e^{-x} - 9x^5e^{-x} - 3x^2y^2e^{-x} - 3x^4e^{-x} + 6y^2e^{-x} + 4x^6e^{-x} + 8x^3y^2e^{-x} + 8y^4e^{-x}}{4\left(x^3+y^2\right)^{\frac{3}{2}}} \\
&= e^{-x}\cdot\frac{6xy^2 - 9x^5 - 3x^2y^2 - 3x^4 + 6y^2 + 4x^6 + 8x^3y^2 + 8y^4}{4\left(x^3+y^2\right)^{\frac{3}{2}}}
\end{aligned}$$

a few errors somewhere, but close enough...

(‡)

$$\begin{aligned}
&\frac{\partial^2}{\partial y\partial x} e^{-x}\sqrt{x^3+y^2} \\
&= \frac{\partial}{\partial y}\left(\frac{3x^2e^{-x}}{2\sqrt{x^3+y^2}} - e^{-x}\sqrt{x^3+y^2}\right) \\
&= \frac{-3x^2ye^{-x}}{2}\cdot\left(x^3+y^2\right)^{-\frac{3}{2}} - \frac{2ye^{-x}}{2\sqrt{x^3+y^2}}
\end{aligned}$$

(††)

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y} e^{-x}\sqrt{x^3+y^2} \\
&= \frac{\partial}{\partial x}\,\frac{-ye^{-x}}{\sqrt{x^3+y^2}} \\
&= \frac{ye^{-x}\sqrt{x^3+y^2} + ye^{-x}\cdot 1/2\cdot\left(x^3+y^2\right)^{-\frac{1}{2}}}{x^3+y^2} \\
&= \frac{ye^{-x}\left(\sqrt{x^3+y^2} + 1/2\cdot\left(x^3+y^2\right)^{-\frac{1}{2}}\right)}{x^3+y^2}
\end{aligned}$$

b)

define

$$f^i_k := \frac{\partial f^i}{\partial y^k} \qquad f^i_{kl} := \frac{\partial f^i}{\partial y^k \partial y^l}$$

and let the jacobian matrix

$$\mathfrak{J} := \left[\left(f_y\right)_{ij} = f^i_j\right]$$

where f_y^2 is the matrix product and f_{yy} has entries $\left(f^i_{kl}\right)$.

consider

$$f : \mathbb{R}^m \to \mathbb{R}^m,\ y \mapsto \left(f^1\left(y^1, ..., y^m\right), ..., f^m\left(y^1, ..., y^m\right)\right)^T$$

show that

$$\left(f_y f\right)_y f = f^T f_{yy} f + f_y^2 f$$

$$y \mapsto f(y) \mapsto \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(f(x))\right)$$

$$\Rightarrow \left(f_y f\right)_y f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(f(f(y)))\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(f^2(y))\right)$$

$$y \mapsto f(y) \mapsto f^T\left(\frac{\partial^2 f}{\partial y^2}(x)\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}(x)\right)$$

$$\Rightarrow f^T f_{yy} f + f_y^2 f = \left(f\left(\frac{\partial^2 f}{\partial x^2}(f(y)) \right) \right)^T + \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}(f(y)) \right)$$

we can see through η-reduction that we only need to show

$$\left(f_y f\right)_y = f^T f_{yy} + f_y^2$$

and further that

$$f_y f = f^T f_y + f_y f$$

thus we need to prove that

$$f^T f_y = 0$$

but

$$f^T f_y = \left(f\left(\frac{\partial f}{\partial y} \right) \right)^T = \left(f^1\left(f_y^1, ..., f_y^m\right), ..., f^m\left(f_y^1, ..., f_y^m\right) \right)$$

no more, i yield, i yield!!

$\boxed{\textbf{problem 2}}$

a)

let $f(x) := x^4 + 3x^3 - 2x + 5$; find all taylor polynomials around $x_0 := -2$.

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recall that each term is given by

$$P_k(x) := \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

for $k \in [0, \deg(f)] \cap \mathbb{Z}$.

first compute

$$\begin{aligned}
f'(x) &= 4x^3 + 9x^2 - 2 \\
f''(x) &= 12x^2 + 18x \\
f^{(3)}(x) &= 24x + 18 \\
f^{(4)}(x) &= 24
\end{aligned}$$

then

$$\begin{aligned}
P_0(x) &= f(-2) = 16 - 24 + 4 + 5 = 1 \\
P_1(x) &= f'(-2) \cdot (x + 2) = 2(x + 2) = 2x + 4 \\
P_2(x) &= \frac{f''(-2)}{2} \cdot (x + 2)^2 = 6x^2 + 24x + 24 \\
P_3(x) &= \frac{f^{(3)}(-2)}{6} \cdot (x + 2)^3 = -5(x + 2)^3 \\
&= -5x^3 - 30x^2 - 60x - 40 \\
P_4(x) &= (x + 2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16
\end{aligned}$$

note that there are finitely many unique derivatives, as $f(x)$ is a polynomial of degree 4.

then the k-th taylor polynomial can be expressed as

$$T_k := \sum_{i=0}^{k} P_i$$

or recursively

$$T_k = T_{k-1} + P_k \quad \wedge \quad T_0 = P_0 = 1$$

for $k \in \mathbb{N}^+$.

thus the taylor polynomials for f are

$$\begin{aligned}
T_0 &= P_0 = 1 \\
T_1 &= P_0 + P_1 = 2x + 5 \\
T_2 &= 6x^2 + 26x + 29 \\
T_3 &= -5x^3 - 24x^2 - 34x - 11 \\
T_4 &= x^4 + 3x^3 - 2x + 5 = f(x)
\end{aligned}$$

naturally we are able to perfectly describe a fourth-degree polynomial with a taylor series.

b)

let $g(x) := \ln(1 + x)$; calculate its maclaurin series – i.e. taylor series at $x_0 = 0$.

---

first we differentiate

$$\begin{aligned}
g'(x) &= \frac{1}{1+x} \\
g''(x) &= -\frac{1}{(1+x)^2} \\
&\vdots \\
g^{(k)}(x) &= (-1)^{k-1} \cdot (k-1)! \cdot (1+x)^{-k}
\end{aligned}$$

for $k \in \mathbb{N}^+$.

as the function is infinitely differentiable and continuous around $x = 0$, we can conclude that $g(x)$ is analytic.

recall that the maclaurin series of an analytic function f is

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

fortunately we have

$$g^{(k)}(0) = (-1)^{k-1} \cdot (k-1)!$$

so

$$\begin{aligned} g(x) &= \sum_{i=1}^{k} \frac{g^{(i)}(0)}{i!} x^i \\ &= \sum_{i=1}^{k} (-1)^{i-1} \frac{(i-1)!}{i!} x^i \\ &= \sum_{i=1}^{k} \frac{(-1)^{i-1} x^i}{i} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \end{aligned}$$

thus we have the maclaurin series of $g(x)$.

problem 3

a)

are $1 + x$, $1 - x$ and $x - x^2$ linearly independent in P_2? what do they span?

let us denote these in vector notation as

$$\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix} \text{ and } \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

respectively. thus we can determine their dependency via gauss-jordan elimination

$$\begin{pmatrix}0 & 0 & -1\\1 & -1 & 1\\1 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = I_3$$

the three vectors are thus linearly independent. then what is their span? since they are linearly independent in P_2, they form a basis for the vector space and thus span out P_2. of note: P_2 is itself isomorphic to $\mathbb{R}^3$.

b)

our affine space has two conditions, $p(1) = 1$ and $p(2) = 2$.

let

$$p(x) := ax^3 + bx^2 + cx + d$$

such that

$$p(1) = a + b + c + d = 1$$

and

$$p(2) = 8a + 4b + 2c + d = 2$$

we can represent this system with a matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 7 & 3 & 1 & 0 & 1 \end{array}\right)$$

thus the condition vectors are linearly independent and the matrix has rank 2, so they span out a two-dimensional constraint space. by the rank-nullity theorem we can conclude that the solution space must have

$$\mathrm{dim}(P_3) - \mathrm{rank} = 4 - 2 = 2$$

dimensions.

c)

we can create a basis out of the three vectors

$$x - 1, x^2 - 1 \text{ and } x^3 - 1$$

to form a three-dimensional space, since they are linearly independent.

from the results of the last task we can intuitively guess that the three conditions will lead to a system of equations with three unknowns. thus the system is solvable and we may indeed choose arbitrary values for our conditions.

so let

$$p(x) := \alpha(x-1) + \beta\left(x^2 - 1\right) + \gamma\left(x^3 - 1\right)$$

such that

$$\begin{aligned} p(0) &= -\alpha - \beta - \gamma = y_0, \\ p(1) &= 0 = y_1, \\ p(2) &= \alpha + 3\beta + 7\gamma = y_2 \end{aligned}$$

this is different from my original expectation. we can tell that y_1 must be 0, thus cannot be chosen arbitrarily. then we effectively only have two equations, thus ending up in the same situation as the last subtask, meaning we will not be able to choose the remaining values arbitrarily, since the system of equations will be underdetermined.

as such, if we choose for $y_1 = 0$ to hold, it is possible.

problem 4

a)

prove that $\{\sin(t), \cos(t), 1\}$ is orthogonal in the space $C[0, 2\pi]$ with inner product

$$\langle f, g \rangle := \int_0^{2\pi} f(s)g(s)\,ds$$

i.e. that it is an orthogonal basis for $C[0, 2\pi]$.

---

we must compute the pair-wise inner product of each base vector

$$\langle \sin, 1 \rangle = \int_0^{2\pi} \sin(t)\,dt = 0,$$
$$\langle 1, \cos \rangle = \int_0^{2\pi} \cos(t)\,dt = 0$$

because $\sin(t)$ and $\cos(t)$ all have a period of 2π. lastly,

$$
\begin{aligned}
\langle \sin, \cos \rangle &= \int_0^{2\pi} \sin(t)\cos(t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} \sin(2t)\,dt \\
&= \frac{1}{4}\int_0^{4\pi} \sin(u)\,du = 0.
\end{aligned}
$$

thus they are all orthogonal and they form a basis under this definition of inner product. to make it an orthonormal basis, we can scale each base component by its length, such that

$$
\mathfrak{O} := \left\{ \frac{\sin(t)}{a}, \frac{\cos(t)}{b}, \frac{1}{c} \right\}
$$

forms an orthonormal basis, where

$$
\begin{aligned}
a &= \sqrt{\langle \sin, \sin \rangle} = \left(\int_0^{2\pi} \sin^2 t\,dt \right)^{1/2} = \sqrt{\pi} \\
b &= \sqrt{\langle \cos, \cos \rangle} = \left(\int_0^{2\pi} \cos^2 t\,dt \right)^{1/2} = \sqrt{\pi} \\
c &= \sqrt{\langle 1, 1 \rangle} = \left(\int_0^{2\pi} dt \right) = \sqrt{2\pi}
\end{aligned}
$$

b)

to form an orthonormal basis for the monomials $\left\{1, x, x^2\right\}$, we use the gram-schmidt method with $\nu_1 := 1$.

then

$$
\begin{aligned}
\boldsymbol{v}_2 &:= x - \mathrm{proj}_{\boldsymbol{v}_1}(x) \\
&= x - \left(\frac{\langle \boldsymbol{v}_1, x\rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle}\right)\boldsymbol{v}_1 \\
&= x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle} \\
&= x - \frac{1}{2}\int_{-1}^{1} x \, dx \\
&= x
\end{aligned}
$$

and similarly for the last vector

$$
\begin{aligned}
\boldsymbol{v}_3 &:= x^2 - \mathrm{proj}_{\boldsymbol{v}_1}\left(x^2\right) - \mathrm{proj}_{\boldsymbol{v}_2}\left(x^2\right) \\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2 \, dx - \left(\int_{-1}^{1} x^2 \, dx\right)^{-1} \cdot \int_{-1}^{1} x^3 \, dx \\
&= x^2 - \frac{1}{3}
\end{aligned}
$$

thus we have found an orthonormal base using gram-schmidt

$$
\mathfrak{O} := \left\{\frac{1}{a}, \frac{x}{b}, \frac{x^2 - \frac{1}{3}}{c}\right\}
$$

where $a = \sqrt{2}$, $b = \sqrt{\frac{2}{3}}$ and $c = \sqrt{\frac{8}{45}}^{\dagger}$.

$$
\begin{aligned}
\dagger\colon \langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle &= \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx \\
&= \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \frac{8}{45}
\end{aligned}
$$