exercise 10

these are my solutions to the tenth exercise set of TMA4135.

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problem 1

a)

$$f(x) := \sin(3x) + 5\cos(2x)$$

is already in its fourier series form.

the coefficients can be read as

$$\begin{aligned} a_0 &= 0, \\ a_n &= \begin{cases} 5 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \\ b_n &= \begin{cases} 1 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

b)

let

$$g(x) = x^2, \quad -\pi \leqslant x \leqslant \pi$$

then we can compute the fourier series using the formulas for 2π -periodic functions

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx \\ &= \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2}{3}\pi^2 \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx \\
&= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_{-\pi}^{\pi} \\
&= \frac{2}{n\pi}\left[x^2\sin(nx) + \frac{2x}{n}\cos(nx) - \frac{2}{n^2}\sin(nx)\right]_{0}^{\pi} \\
&= \frac{2}{n\pi}\left(\frac{2\pi}{n}\cos(\pi n)\right) \\
&= \frac{4}{n^2}(-1)^n
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx \\
&= \frac{1}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\left(\cancel{-\frac{\pi^2}{n}\cos(\pi n)} + \frac{2\pi}{n^2}\sin(\pi n) + \cancel{\frac{2}{n^3}\cos(\pi n)}\right)\right. \\
&\left.-\left(\cancel{-\frac{\pi^2}{n}\cos(-\pi n)} - \frac{2\pi}{n^2}\sin(-\pi n) + \cancel{\frac{2}{n^3}\cos(-\pi n)}\right)\right] \\
&= \frac{2}{n^2}(\sin(\pi n) + \sin(-\pi n)) \\
&= \frac{2}{n^2}(\sin(\pi n) - \sin(\pi n)) = 0
\end{aligned}
$$

the fact that b_n is zero is reassuring, since the oddness that it contributes with would be undesirable for a function like x^2.

c)

show

$$h(x) = f(x) * g(x) = \int_{-\pi}^{\pi} g(y) f(x-y)\, dy$$
$$= -\frac{4\pi}{9} \sin(3x) + 5\pi \cos(2x)$$

$$h(x) = \int_{-\pi}^{\pi} y^2 [\sin(3(x-y)) + 5\cos(2(x-y))]\, dy$$

this would be an awful lot of integration by parts, so let's be more clever about this.

we can expand the trigonometric terms in $f(x-y)$ to obtain simpler terms that cancel out under $\int_{-\pi}^{\pi} g(y) \sin(ny)\, dy = 0$

- $\sin(3(x-y)) = \sin(3x)\cos(3y) - \cos(3x)\sin(3y)$
- $\cos(2(x-y)) = \cos(2x)\cos(2y) + \sin(2x)\sin(2y)$

thus we obtain

$$\begin{aligned} h(x) = \sin(3x) &\int_{-\pi}^{\pi} y^2 \cos(3y)\, dy \\ &\cancel{-\cos(3x) \int_{-\pi}^{\pi} y^2 \sin(3y)\, dy} \\ &+5\cos(2x) \int_{-\pi}^{\pi} y^2 \cos(2y)\, dy \\ &\cancel{+5\sin(2x) \int_{-\pi}^{\pi} y^2 \sin(2y)\, dy} \end{aligned}$$

but these are simply the coefficients of y^2 from b), such that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} y^2\cos(2y)\,dy = \left[\frac{4}{n^2}(-1)^n\right]_{n=2} = 1$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} y^2\cos(3y)\,dy = \left[\frac{4}{n^2}(-1)^n\right]_{n=3} = -\frac{4}{9}$$

which gives

$$h(x) = -\frac{4\pi}{9}\sin(3x) + 5\pi\cos(2x)$$

∎

d)

$h(n)$ is also made up of basis vectors, such that the coefficients can be written as

$$a_0 = 0$$
$$a_n = \begin{cases} 5\pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$
$$b_n = \begin{cases} -\frac{4\pi}{9} & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$c_n(f) = a_n(f) + b_n(f) = \begin{cases} 5 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$c_n(g) = \begin{cases} \frac{2}{3}\pi^2 & \text{if} \quad n = 0 \\ \frac{4}{n^2}(-1)^n & \text{otherwise} \end{cases}$$

$$c_n(h) = \begin{cases} 5\pi & \text{if} \quad n = 2 \\ -\frac{4\pi}{9} & \text{if} \quad n = 3 \\ 0 & \text{otherwise} \end{cases}$$

we can notice that $c_2(h) = \pi c_2(f)$ and $c_3(h) = \pi c_3(g)$.

if we look for a deeper pattern, we can also notice that $c_2(g) = 1$ and $c_3(f) = 1$ such that we could say that

$$c_2(h) = \pi c_2(f) c_2(g) \quad \wedge \quad c_3(h) = \pi c_3(f) c_3(g)$$

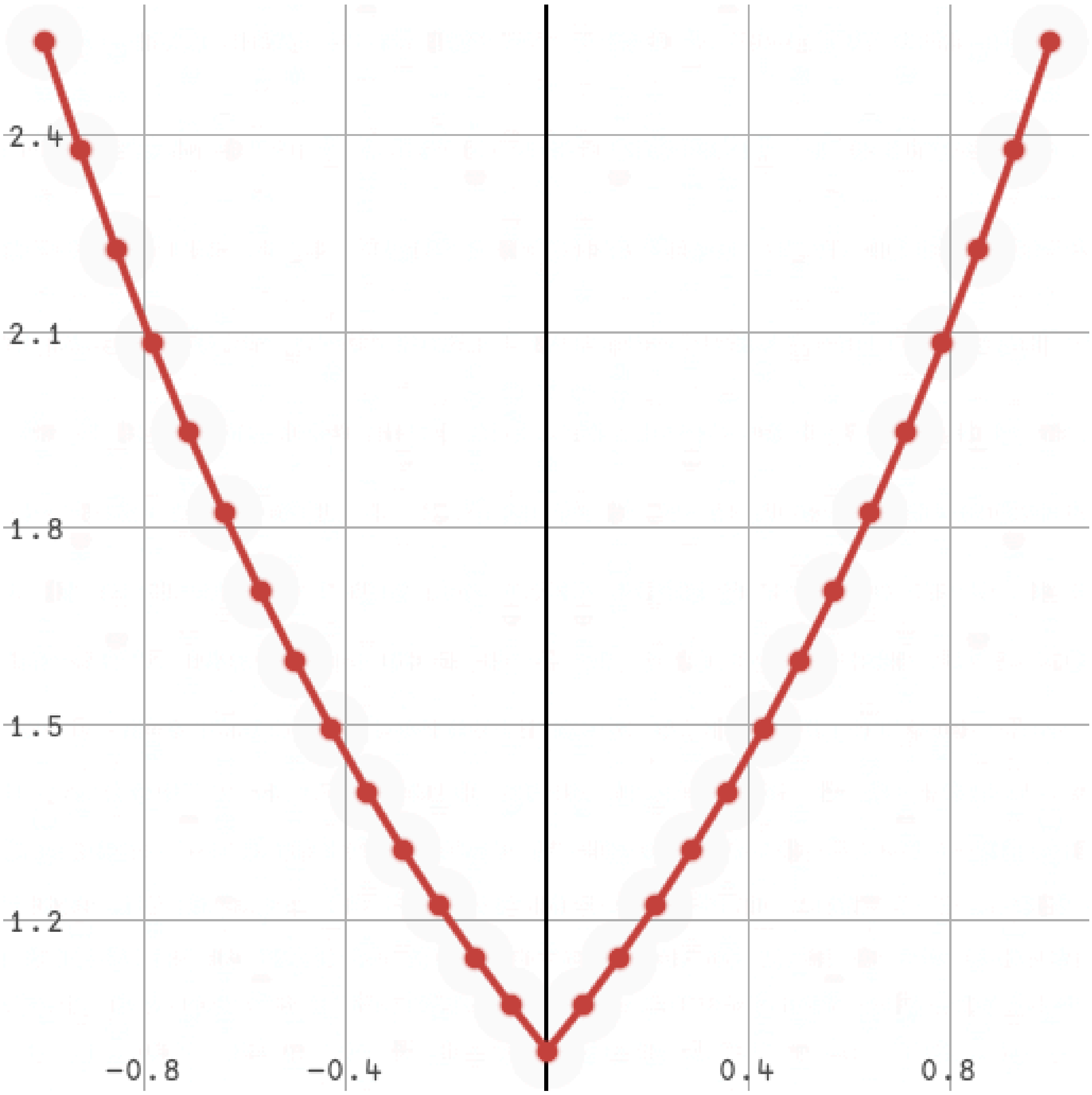
this is a profound result – well, the generalized statement is – and is related to the fourier transform. we are saying that convolving two functions is the same as taking their point-wise product at certain frequencies.

problem 2

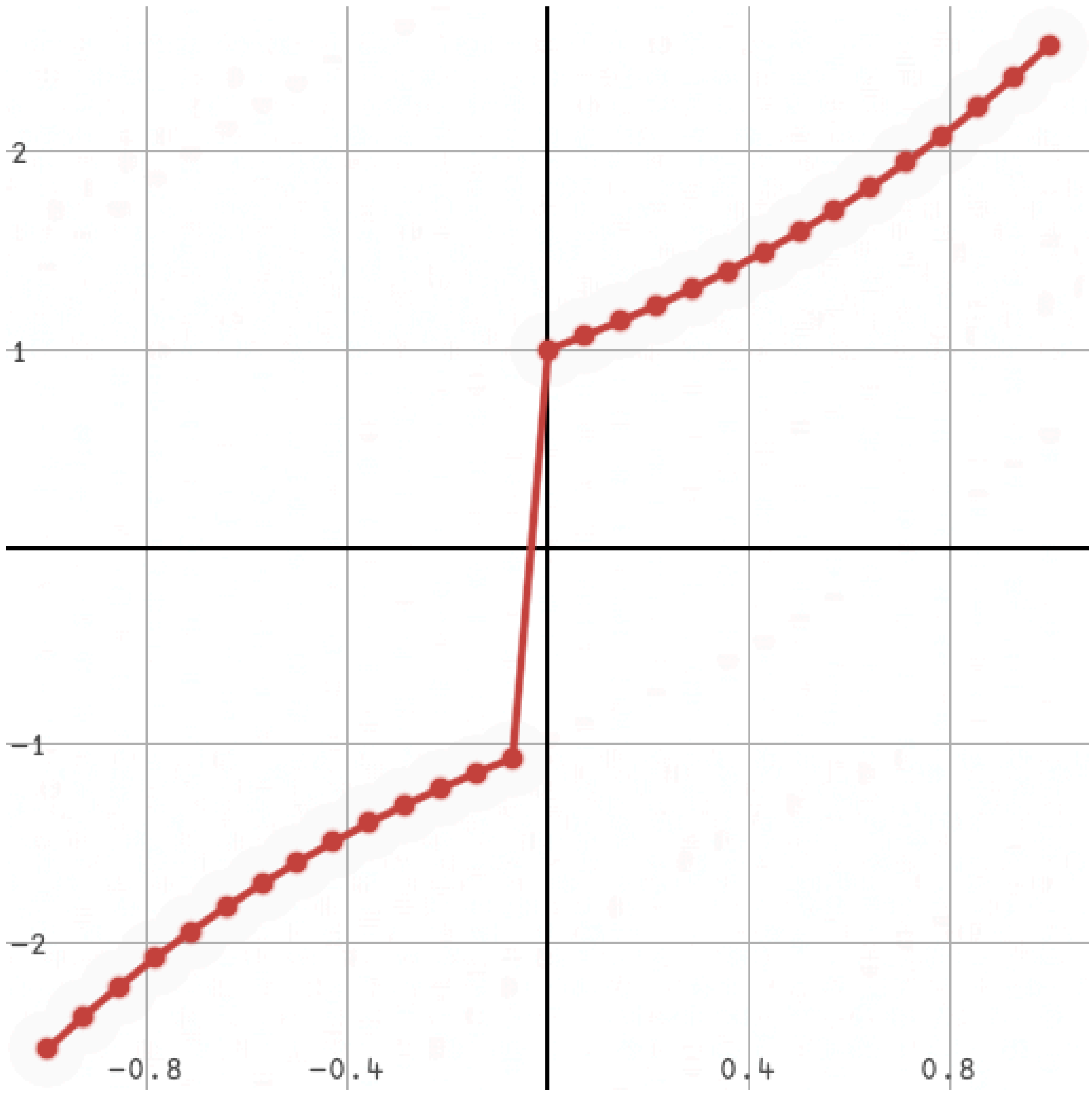
a)

click the code to obtain the sketches for the expansions of $f(x)$.

```
# Experimental!
~ "git: github.com/Omnikar/uiua-plot" ~ Data Plot
∩⌟⊂⊸¯⇌⊸↘1 ₑ÷⟜⇡15
∩⌞⊟-1×2÷-1⊸⧻°⊏
∩(Plot Data ⍉)
```



this is how the even expansion would look.



this is how the odd expansion would look. it is a little off because i didn't bother to make a duplicate point at -1 to highlight the discrete jump.

these are both created by evaluating $f(x) = \exp(x)$ on $[0, 1]$, then taking those values in reverse from 0 to -1, one of them negated to obtain the odd property.

b)

it makes sense that the even expansion is more practical in this case, since it is smoother and doesn’t contain a discrete jump, only a point at which it isn’t differentiable, but it is continuous.

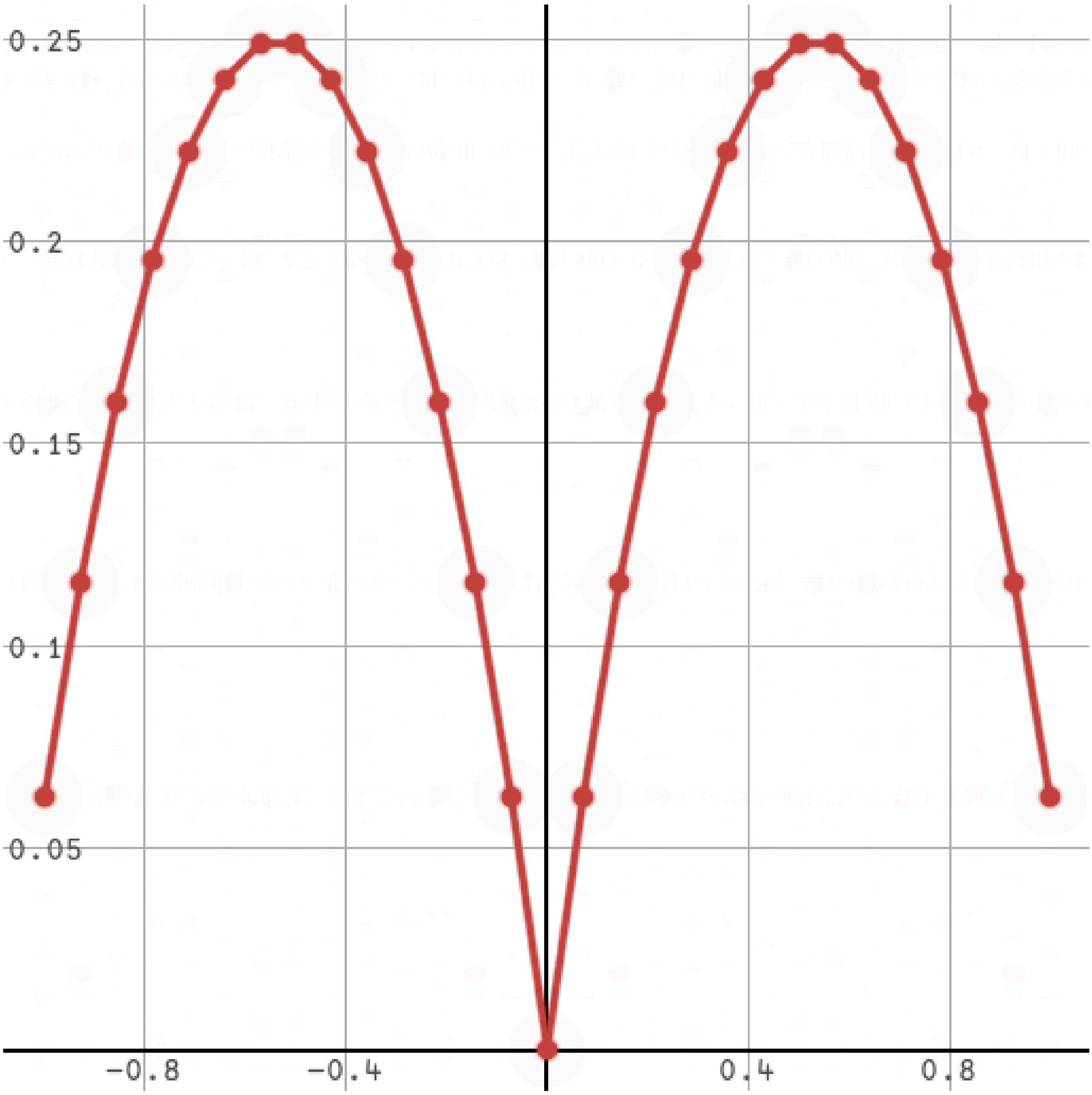
the gibbs artifacts will be smaller compared to the odd expansion, which will have large artifacts at the discrete jump, making for a less accurate approximation of $f(x)$ using a truncated fourier series.

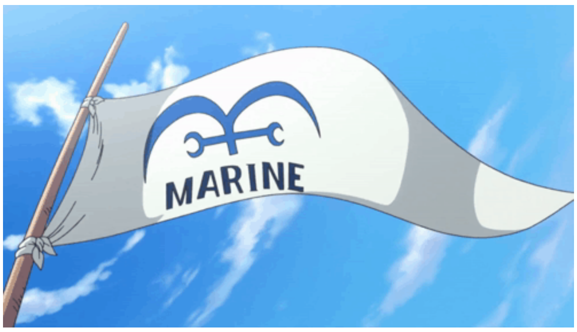
c)

let now $f(x) = x - x^2$ on $[0, 1]$.

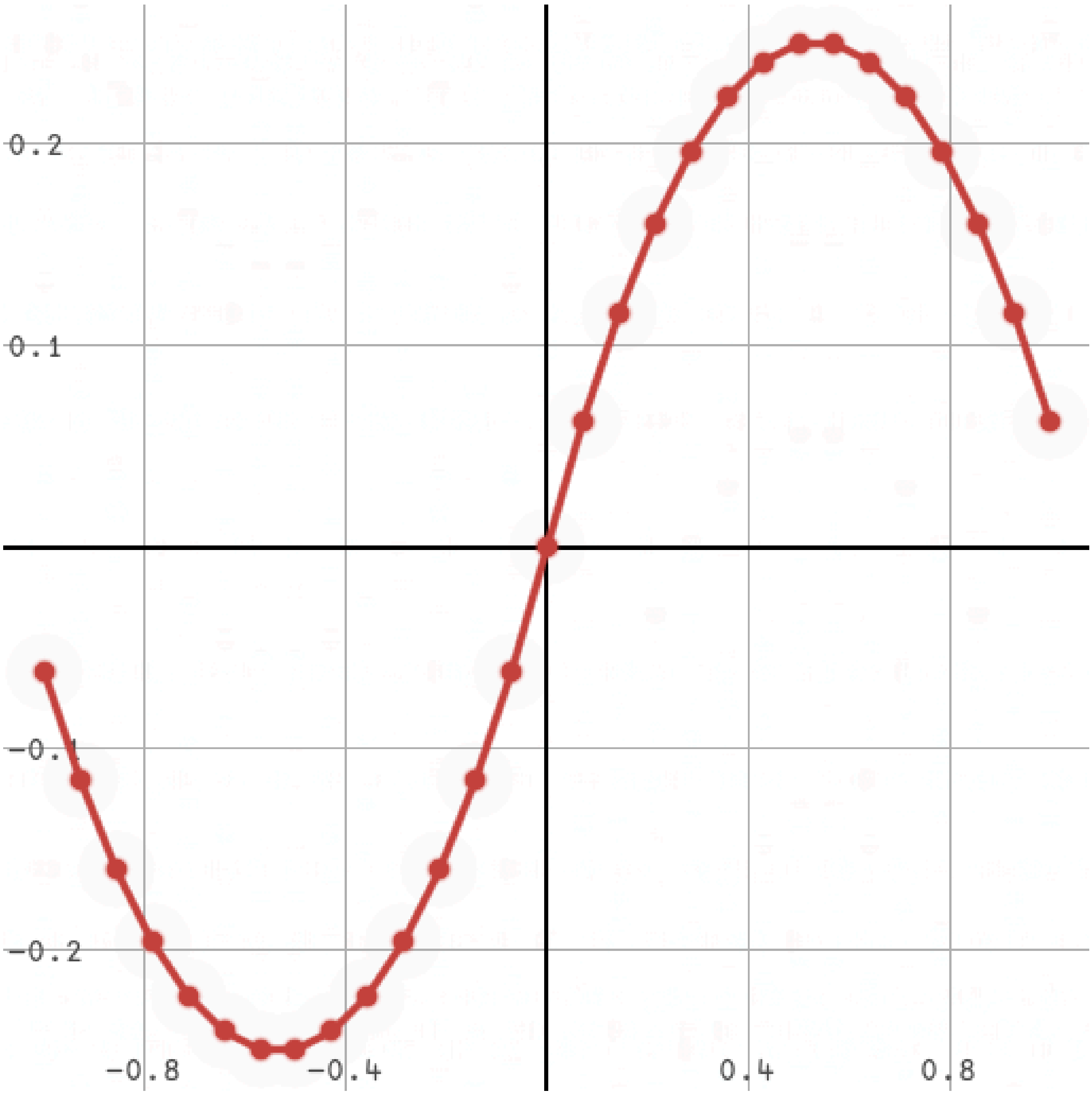
we only need to change our single function in the code, namely change from `ₑ` to `(-⊸°√)`.

```
... ∩⌟⊂⊸¯⇌⊸↘1 (-⊸°√)÷⟜⇡15 ...
```





this is how the even expansion looks.



this is the odd expansion. very nice.

we can see that the odd expansion already resembles a sine-wave. we can then guess that this will the more accurate one of the two.

d)

denote the even and odd expansions as functions respectively

$$g(x) = \begin{cases} x - x^2 & \text{if} \quad x \geqslant 0 \\ -x - x^2 & \text{otherwise} \end{cases}, \quad h(x) = \begin{cases} x - x^2 & \text{if} \quad x \geqslant 0 \\ -x + x^2 & \text{otherwise} \end{cases}$$

we can find the coefficients for these by calculating the integrals from the definitions of the coefficients

$$\begin{aligned}
a_0(g) &= \int_{-1}^{1} g(x)\,dx \\
&= \int_{-1}^{0} \left(-x - x^2\right) dx + \int_{0}^{1} \left(x - x^2\right) dx \\
&= -\frac{1}{2}\left[x^2\right]_{-1}^{0} - \frac{1}{3}\left[x^3\right]_{-1}^{0} + \frac{1}{2}\left[x^2\right]_{0}^{1} - \frac{1}{3}\left[x^3\right]_{0}^{1} \\
&= -\frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = -\frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n(g) &= \int_{-1}^{1} g(x)\cos(2\pi n x)\,dx \\
&= \int_{-1}^{0} \left(-x - x^2\right)\cos(2\pi n x)\,dx + \int_{0}^{1} \left(x - x^2\right)\cos(2\pi n x)\,dx \\
&= 2\int_{0}^{1} \left(x - x^2\right)\cos(2\pi n x)\,dx \\
&= 2\left[\frac{x - x^2}{2\pi n}\sin(2\pi n x) + \frac{1 - 2x}{4\pi^2 n^2}\cos(2\pi n x) + \frac{2}{8\pi^3 n^3}\sin(2\pi n x)\right]_{0}^{1} \\
&= 2\left[-\frac{1}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] \\
&= -\frac{1}{\pi^2 n^2}
\end{aligned}$$

$$b_n(g) = \int_{-1}^{1} g(x)\sin(2\pi n x)\,dx = 0$$

$$
\begin{aligned}
a_0(h) &= \int_{-1}^{1} h(x)\,dx \\
&= \int_{-1}^{0}\left(-x + x^2\right)dx + \int_{0}^{1}\left(x - x^2\right)dx \\
&= -\frac{1}{2}\left[x^2\right]_{-1}^{0} + \frac{1}{3}\left[x^3\right]_{-1}^{0} + \frac{1}{2}\left[x^2\right]_{0}^{1} - \frac{1}{3}\left[x^3\right]_{0}^{1} \\
&= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = 0
\end{aligned}
$$

$$
a_n(h) = \int_{-1}^{1} h(x)\cos(2\pi n x)\,dx = 0
$$

$$
\begin{aligned}
b_n(h) &= \int_{-1}^{1} h(x)\sin(2\pi n x)\,dx \\
&= \int_{-1}^{0}\left(-x + x^2\right)\sin(2\pi n x)\,dx + \int_{0}^{1}\left(x - x^2\right)\sin(2\pi n x)\,dx \\
&= 2\int_{0}^{1}\left(x - x^2\right)\sin(2\pi n x)\,dx \\
&= 2\left[\frac{x^2 - x}{2\pi n}\cos(2\pi n x) + \frac{1 - 2x}{4\pi^2 n^2}\sin(2\pi n x) - \frac{2}{8\pi^3 n^3}\cos(2\pi n x)\right]_{0}^{1} \\
&= 0
\end{aligned}
$$

well, that's definitely wrong.

problem 3

a)

the truncated fourier series of a function looks like

$$f_N = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx)$$

note: i use $\frac{a_0}{2}$ with a $\frac{1}{L}$ in the definition of a_0.

then to calculate the error, we take the difference with the actual function, which can be expressed as an infinite fourier series

$$\begin{aligned}
e_N(x) &= f(x) - f_N(x) \\
&= \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)\right] \\
&\quad - \left[\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx)\right] \\
&= \sum_{n=N+1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))
\end{aligned}$$

it is the tail end of the infinite sum.

parsevals identity states that

$$\langle f, f \rangle = \frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

b)

let $f(x) = e^{-x}$

we first need to find the fourier coefficients

$$a_0 = \int_{-1}^{1} e^{-x}\,dx = [e^x]_{-1}^{1} = e - \frac{1}{e}$$

$$a_n =$$

problem 4

a)

$$f(x) = \begin{cases} -\pi - x & \text{if} -\pi < x < -\frac{\pi}{2} \\ x & \text{if} -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \leqslant \pi \end{cases}$$

$f(x)$ is odd, so $a_0 = a_n = 0$.

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx \\ &= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} x \sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin(nx)\,dx \right] \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin(nx)\,dx &= \left[-\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_0^{\frac{\pi}{2}} \\ &= -\frac{\pi}{2n} \cos\Big(n\frac{\pi}{2}\Big) + \frac{1}{n^2} \sin\Big(n\frac{\pi}{2}\Big) \end{aligned}$$

$$\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,dx = (-1)^n \int_0^{\frac{\pi}{2}} u\sin(nu)\,du$$

$$= (-1)^n \left[-\frac{\pi}{2n}\cos\left(n\frac{\pi}{2}\right) + \frac{1}{n^2}\sin\left(n\frac{\pi}{2}\right)\right]$$

thus

$$b_n = \frac{2}{\pi}\left(1 + (-1)^n\right)\left[-\frac{\pi}{2n}\cos\left(n\frac{\pi}{2}\right) + \frac{1}{n^2}\sin\left(n\frac{\pi}{2}\right)\right]$$

for even n: $\cos\left(n\frac{\pi}{2}\right) \in \{-1, 0, 1\}$ and $\sin\left(n\frac{\pi}{2}\right) = 0$, so $b_n = 0$

for odd n: $\cos\left(n\frac{\pi}{2}\right) = 0$ and $\sin\left(n\frac{\pi}{2}\right) = (-1)^{\frac{n-1}{2}}$

$$b_n = \frac{4}{\pi n^2}\sin\left(n\frac{\pi}{2}\right)$$

for $n = 2k + 1$:

$$b_{2k+1} = \frac{4}{\pi(2k+1)^2}(-1)^k$$

applying parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2$$

$$\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x^2\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)^2\,dx\right] = \frac{2}{\pi}\left[\frac{\pi^3}{24} + \frac{\pi^3}{24}\right] = \frac{\pi^2}{12}$$

$$\sum_{k=0}^{\infty} \frac{16}{\pi^2(2k+1)^4} = \frac{\pi^2}{12}$$

so

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

b)

even extension of $f(x) = x$ on $[0, 1]$ is $f(x) = |x|$ with period 2.

$$\begin{aligned}
a_0 &= 2\int_0^1 x\,dx = 1 \\
a_n &= 4\int_0^1 x\cos(n\pi x)\,dx \\
&= 4\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_0^1 \\
&= \frac{4}{n^2\pi^2}\big((-1)^n - 1\big)
\end{aligned}$$

$a_n = 0$ for even n

$a_n = -\frac{8}{n^2\pi^2}$ for odd n

the series is

$$f(x) = \frac{1}{2} - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x)$$

evaluating at $x = 0$ gives $f(0) = 0$:

$$0 = \frac{1}{2} - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

therefore

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{16}$$

c)

odd extension of $f(x) = x$ on $[0, 1]$ is $f(x) = x$ with period 2.

$$\begin{aligned} b_n &= 4 \int_0^1 x \sin(n\pi x)\, dx \\ &= 4 \left[-\frac{x}{n\pi} \cos(n\pi x) + \frac{1}{n^2\pi^2} \sin(n\pi x) \right]_0^1 \\ &= -\frac{4}{n\pi}(-1)^n = \frac{4}{n\pi}(-1)^{n+1} \end{aligned}$$

the can be written as

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)$$

evaluating at $x = \frac{1}{2}$:

$$\frac{1}{2} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(n\frac{\pi}{2}\right)$$

since $\sin\left(n\frac{\pi}{2}\right) = 0$ for even n and $\sin\left((2k+1)\frac{\pi}{2}\right) = (-1)^k$ for odd n:

$$\frac{1}{2} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}(-1)^k = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1}$$

therefore

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$