assignment 2
tdt4125 - algorithm construction

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task 1

this sounds like the vertex cover problem, but with additional vertex weights to minimize.

a) recall the general key components of a linear program

- problem input vector x
- coefficient vector c
- constraint matrix A
- target vector b

such that we minimize $c^\mathsf{T} x$ with respect to $Ax \geq b$.

here we can interpret the input vector x as a binary vector where the i-th bit encodes the inclusion of a given node $v_i \in V$ in the subset C. x has as many bits as there are vertices.

furthermore, the vector c represents the weights of the vertices such that

$$c^\mathsf{T} x = \sum_{v \in V} c_v x_v = \sum_{v \in C} c_v$$

becomes the minimization goal.

we can express the binary string x through constraints

- $x_u + x_v \geq 1 \quad \forall (u, v) \in E$
- $x_v \in \{0, 1\} \quad \forall v \in V$

the former constraint ensures that every edge is covered by at least one endpoint.

bringing it all together

$$\begin{aligned} &\min \sum_{v \in V} c_v x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall (u, v) \in E \\ & x_v \in \{0, 1\} \quad \forall v \in V \end{aligned}$$

is our complete linear integer program.

we have implicitly defined A to be the $|E| \times |V|$-matrix that has 1-entries for vertices v where $e = (u, v)$. it is zero otherwise.

b) by relaxing the linear integer program we let $0 \leq x_v \leq 1$. this can be shortened to saying $x_v \geq 0$ since the first constraint forces $x_v \leq 1$. requiring a positive value is a common restriction on linear programs and allows us to take the dual.

we then obtain the dual program

$$\begin{aligned} &\max \mathbf{1}^\mathsf{T} y = \sum_{e \in E} y_e \\ &\text{s.t.} \quad A^\mathsf{T} y \leq c \\ &\Leftrightarrow \sum_{v \in e} y_e \leq c_v \\ &\forall v \in V, \quad y \geq 0 \end{aligned}$$

c) to construct a primal-dual algorithm to approximate this problem, we use the formulations given previously.

Algorithm 1: Primal-Dual Weighted Vertex Cover Approximation

```
procedure VERTEX-COVER(G, c)
    ▷ initialize primal (x) and dual (y) variables
    for v ∈ V, e ∈ E do
        y_e ← 0
```

$x_v \leftarrow 0$
end

while there exists an uncovered edge $e = (u, v)$ **do**
▷ calculate minimum slack
$\text{slack}_u \leftarrow c_u - \sum_{e' \in \delta(u)} y_{e'}$
$\text{slack}_v \leftarrow c_v - \sum_{e' \in \delta(v)} y_{e'}$
$\alpha \leftarrow \min(\text{slack}_u, \text{slack}_v)$

▷ increase dual variable for edge e
$y_e \leftarrow y_e + \alpha$

▷ Update primal if constraint becomes tight
if $\sum_{e' \in \delta(u)} y_{e'} = c_u$ **then**
$x_u \leftarrow 1$
end
if $\sum_{e' \in \delta(v)} y_{e'} = c_v$ **then**
$x_v \leftarrow 1$
end
end
return $C = \{v \in V \mid x_v = 1\}$
end

where $\delta(u)$ means the set of all edges going out of vertex u.

other

the rest of the tasks are attached or can be found at https://git.pvv.ntnu.no/frero-uni/TDT4125/src/branch/main/assignment-2